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On the dynamics of a SIR model for a financial risk contagion

Mauro Aliano[1] · Lucianna Cananà[2] · Tiziana Ciano[3] · Stefania Ragni[1] · Massimiliano Ferrara[4]

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Abstract
This work starts from an analogy between financial systems and ecosystems so that the SIR mathematical approach can be revisited in modeling a kind of risk contagion among financial players. We are interested on a specific type of financial risk contagion which identifies firms as the key participants responsible for propagating this contagion. In this respect, the proposed mechanism facilitating this transmission is the Supply Chain framework. In this direction, we focus on a new SIR dynamic with time delay which represents the "financial immunity" after recovery. A complete and robust analysis about asymptotic stability is performed for both risk-free and not-free-risk steady states at the long run, by applying Lyapunov functional method. The model is applied to perform some simulations with application in different Italian economic sectors.

Keywords SIR model · Risk contagion · Financial immunity · Supply chain finance · Asymptotic stability

1 Introduction

"The king, Don Carlos, has escaped the vigilance of his guardians at Bourges, and has returned to Spain by the Catalonian frontier. Barcelona has risen in his favor." Edmond Dantes takes one of his vengeances in "The Count of Monte Cristo" by causing financial harm to the banker Danglars through the dissemination of this news, which turns out to be false later on. We are in France at the beginning of the nineteenth century, and the news is spread by means of an innovative device that is the telegraph. Danglars sells everything, expecting the value of Spanish bonds to fall, resulting in an immediate loss and then a lost profit, due to the fact that the news turns out to be false. Although it occurs in a different manner, the spread of contagion in finance employs communication tools which utilise a scheme similar to that seen in Dumas' novel, while more advanced and modern than the telegraph.

The most well-known instances of financial market contagion are those involving Lehman Brothers, the subprime mortgage crisis, and the European sovereign debt crisis.

Mauro Aliano, Lucianna Cananà, Tiziana Ciano and Stefania Ragni have contributed equally to this work.

Extended author information available on the last page of the article

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The currency market and cryptocurrencies have been the subject of several noteworthy examples. As a notable example, we quote the financial contagion occurred in July 1997, when the Thai currency crisis spread not just East Asia but also Russia and Brazil. Long-Term Capital Management, i.e. a U.S. hedge fund, collapsed as a result of the currency crisis, which spread like an epidemic to other economies.

Within the context of cryptocurrency markets, there was yet another instance of risk spreading. Actually, Three Arrows Capital (3AC) crashed in June 2022 as a result of the so-called margin call failure, and Voyager Digital-the cryptocurrency broker that promoted the 3AC bankruptcy action-also crashed a few weeks later in July 2022.

In this framework, contagion is a key concept in the financial literature, where both endogenous and exogenous events can cause a large chain reaction of crises (Haldane and May 2011). As a result of the domino effect, if one bank (or another financial intermediary) enters a crisis or pre-crisis state, other banks may also enter a crisis or pre-crisis state. According to Callon (1990), network analysis has a long history in economic studies, but it can also be used to explain financial crises. Among the various possibilities, as deeply discussed in Aliano et al. (2023), the literature can be divided: (i) into infections that occur in the banking system, (ii) infections that occur on financial markets, (iii) infections that occur between credit and debit relationships between companies. In particular, this paper focuses on the third item. In this regard, we recall some contributions in the literature. For instance, the analysis in Egloff et al. (2007) deals with the credit deterioration contagion mechanism and states that a company's credit deterioration may also deteriorate credit in other counterparties. In Hertzel et al. (2008) the authors investigate intra-industry bankruptcy and evaluate the effects of distress on both customers and suppliers: they demonstrate the impact of financial wealth on stock price reactions to distress and failures. In supply chain finance, Xie et al. (2023) focuses on a dual-channel financing model that includes bank loans and manufacturer trade credit. According to the authors, credit risk is a contagion channel for Small and Medium Enterprises in the supply chain (SMEs). As the last citation, we recall that in Calabrese (2022) the author investigates the spread of UK small business failures and finds that geographic location is important.

1.1 Risk contagion in supply chain finance

We focus on financial risk infections which occur between credit and debit relationships among companies, as previously mentioned. In particular, we are interested in a specific type of financial risk contagion which identifies firms as the key participants responsible for propagating this contagion. In this respect, the actual mechanism facilitating this transmission is the Supply Chain framework. In order to establish this perspective, we draw upon previous studies in the related literature as a foundation. Over the past decade, the field of supply chain finance research has experienced significant growth and evolution. Notably, the studies carried out in Chakuu et al. (2019), Jia et al. (2020), Wu et al. (2019) and Zhao and Huchzermeier (2015) have systematically investigated the methodologies, implementation strategies, and the resultant impact of this research. The results in the whole literature shed light on the critical role of supply chain linkages as a primary conduit for the transmission of credit risk contagion among counterparties.

The contagion effect of credit risk is a topic of empirical exploration within the domains of finance and management literature. In this respect, various aspects are encompassed such as borrowing costs (see Houston et al. 2016), growth options (see Gamba and Saretto 2020), stock prices (see Filbeck et al. 2016), and subsequent failures (see Jacobson and

Von Schedvin 2015). Additionally, Lian (2017) employes the Z-score index to investigate the impact of supply chain relationships on the financial distress of suppliers, revealing a strong correlation between the financial well-being of key customers and the risk of suppliers encountering financial difficulties. With a specific focus on the consequences of the Covid-19 outbreak, Agca et al. (2023) examines how regional economic disruptions have influenced global supply chains and corporate credit risk. This paper delves into the effects of shocks in a given region on the interactions between businesses, regional suppliers, and customers. A particular emphasis is placed on evaluating the impact of the Covid-19 pandemic on corporate credit risk, with a specific focus on supply chain relationships between the United States and China. Furthermore, Zhang et al. (2019) proposes a risk assessment model to measure credit risk in supply chain finance using a modified Kealhofer Merton Vasicek model and Copula function. We also point out that Filbeck et al. (2016) uses typical event research methodology to examine the effects of supply chain disruptions on the automobile industry; this includes analyzing stock market movements, evaluating daily stock returns, and comparing actual returns to expected returns.

1.2 Our contribution

Our contribution is based on the idea that financial risk spreads in a market like an epidemic spreads among a population in a given ecosystem (see May et al. 2008). In this respect, the epidemiological approach of a Susceptible-Infected-Recovered (SIR) dynamics has been adopted in the literature for modelling risk contagion. For instance, we quote some contribution to the literature: Cao and Zhu (2012) and Fanelli and Maddalena (2020) apply the SIR perspective to describe crisis contagion for the banking network, Garas et al. (2010) studies contagion dynamics and financial crisis among various countries in the world by a SIR perspective, Zhao et al. (2021) exploits this epidemiological approach for the credit risk contagion of Internet peer-to-peer lending platforms, Aliano et al. (2023) and Aliano et al. (2022) analyse some SIR models for risk contagion among economic or financial players.

In this framework, we are interested in describing risk contagion among players or agents in a given economic sector in the framework of SIR dynamics. As a major assumption, a population of economic players is divided into a set of distinct compartments, which are defined in terms of risk with low and high level. The individuals characterized by low risk are recognized as susceptible to contagion when they come into contact with agents in the infected compartment. On the other hand, infectious individuals are identified with players characterized by high risk and can spread the risk infection. Furthermore, recovered agents become able to keep their risk at a low level, after contagion, so that they are no longer infectious for a period of time but not life long. Actually, once the period of financial immunity has expired, recovered players revert to be susceptible to risk contagion again.

The model consists of a time delay differential system and is inspired by the study developed in Kyrychko and Blyuss (2005) for describing a disease transmission and an epidemic behaviour. We apply this approach in the different framework of risk transmission through a given economic sector. As an original issue with respect to the analysis in Kyrychko and Blyuss (2005), we adopt an assumption which is natural from a financial point of view: precisely, we suppose that a portion of players indefinitely leaves the market at an elimination rate which depends on the risk level of compartments they belong to. Therefore, we consider the mortality rate as a parameter which is not uniform, but it changes according with the risk level so that a lower rate is related to lower risk in the economic sector and a higher rate corresponds with higher risk. We notice that, although SIR models are not a novelty, the paradigm is new and its application is worthy of attention.

The mathematical framework is described in detail in Sect. 2. In particular, we state its well-posedness according with the main results of delay differential equation theory (see Diekmann et al. 2012; Richard 2003). In addition, in Sect. 2.1 we point out that the model is characterized by two equilibrium points: the first one corresponds to a risk-free trivial steady state, while the second equilibrium is not-free-risk and represents an endemic state. We point out that risk dynamics is affected by various factors in the model at hand and its evolution becomes complex due to its non linearity. Actually, as a topic that deserves special attention, we are interested in arguing whether the risk continues to exist or it is eliminated from the economic sector in the long run. With this aim, we focus on the stability analysis of both steady states characterizing risk dynamics. To the best of our knowledge, the results we provide are original. In particular, a complete analysis of asymptotic stability concerning the risk-free equilibrium is carried out in Sect. 3. On the other hand, we study the behaviour of the variables at hand around the not-free-risk steady state in Sect. 4, where some sufficient conditions for asymptotic stability are stated by employing Lyapunov functional approach.

This paper, as previously discussed, focuses on the mechanism that allows financial risk transmission inside the supply chain framework. Therefore, in Sect. 5, the SIR model is run on some simulations related to risk contagion in a given supply chain based on data from AIDA-BUREAU VAN DIJK collected at the company level over the last 10 years. In particular, we concentrate on companies operating in two distinct production sectors with differing supply chain characteristics: the food sector in the Emilia Romagna region and the automotive sector in Italy. As of information obtained from the Bank of Italy (2023),[1] the food sector holds a pivotal economic role across the northern regions of Italy. Additionally, the automotive sector boasts one of the most extensive and dynamic manufacturing supply networks, operating in an exceptionally competitive environment, surpassing competitiveness levels found in many other industries (see Colombari et al. 2023). It also plays a pivotal role in contributing to the Italian Gross Domestic Product. Our analysis holds the potential to make meaningful contributions to the existing literature. We aim to shed light on how distinct supply chain characteristics—affected by various productive sectors and variations in geographical scale, such as region versus state—may lead to different policy implications and actionable insights after credit risk spread in companies' relationship and supply chain. Regardless, we would like to remark that the methodology presented in our paper can be employed to assess the risk of contagion in sectors and within national and international economies other than those investigated in our research.

Actually, our analysis might prove valuable in various other productive industries and for assessing the potential spread of credit risk among companies operating within a particular sector. Our conjecture is that a financially troubled company could propagate credit risk throughout the sector via its supply chain. This theory could be further explored by leveraging financial data from intermediaries like banks, enhancing the resilience of financial ties among companies.

[1] Bank of Italy (2023). Regional Economies, https://www.bancaditalia.it/pubblicazioni/economie-regionali/ricerca/ricerca.html?categoria=ecoreg&topicSelect=analisiPerRegioniTopic
The website has been visited on September 21, 2023.

Concluding discussion is drawn in Sect. 6, where we focus on the implications for policy and the functioning of decision-makers.

2 SIR dynamics: elimination rate dependent on risk level

With the aim of modelling risk contagion by SIR dynamics, the players in a given economic sector are classified according to their risk level. As already disclosed in the previous Sect. 1, economic agents are divided in different compartments, as follows.

- Low risk levels define susceptible companies, yet they are susceptible to infection by high risk firms, increasing the likelihood that they will turn into high risk companies.
- Infected companies are contagious due to the fact that they are characterized by a very high level of risk and have the potential to infect other companies with a lower level of risk.
- Companies that are able to control their risk following infection, maintain low levels of risk, and are not contagious are referred to as recovered companies.

We would like to point out that companies of all sizes are considered to be at the same degree of risk, since this classification into compartments based only on risk level ignores the size of individual enterprises. This fundamental assumption, which is typical in the SIR modeling literature, does not limit the applicability of the SIR model in our simulation of financial risk contagion throughout a supply chain encompassing small and medium-sized firms, as detailed in the following Sect. 5.

In order to describe the risk diffusion under a mathematical point of view, at any time t, we denote the densities of susceptible players, infected and recovered agents by $S(t)$, $I(t)$ and $R(t)$, respectively. The whole dynamics of contagion is sketched in Fig. 1.

Firstly, we turn our attention to contagion and remark that this phenomena may be modeled within the context of financial risk by utilizing a basic mathematical method to population dynamics. Indeed, it is possible to think of susceptible companies as prey species and infected ones as predators. This makes it possible to analyze risk contagion using the well-known Holling's response functions, which are a popular tool for examining predator–prey interactions (see Holling 1959a, b; Dawes and Souza 2013) or the removal of invasive species from a given ecosystem (see Baker et al. 2018; Marangi et al. 2020, 2023). Here we adopt the type I response and employ a bilinear incidence term of the form $a\,S(t)I(t)$, where $a > 0$ measures the removal rate due to contagion. As a difference with respect to the Holling's response function of type II, the employment of the previous bilinear incidence term assumes we do not

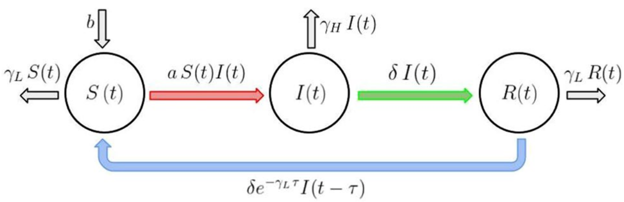

Fig. 1 Scheme of risk contagion dynamics: contagion and healing are highlighted in red and green, respectively; while the reverting from recovered to susceptible is colored in blue. (Colour figure online)

take into account the amount of time an at-risk item needs to incubate and process a susceptible item. This viewpoint makes sense instead of being a modelling limitation since actual data cannot be used to measure the amount of time required for financial risk to incubate; nevertheless, it is reasonable to suppose that the period of risk contagion incubation is so brief as to be regarded as negligible in the time scale of numerical simulations in the next Sections.

While this is going on, some of the infected companies associated with the term $\delta I(t)$ depart from the same class of infectious agents and recover. We suppose $0 < \delta < 1$. Contagious enterprises in the economic sector take the appropriate steps to become immune at a rate indicated by the contagion elimination rate δ, which simulates a cure action. Regardless, we presume that following healing, recovered companies achieve a temporary (but non-permanent) financial immunity ability for a time period of length $\tau > 0$. Actually, once the firms have recovered from an exceptionally high level of risk at time $t - \tau$, they stay in a sort of recovery pool until time t, for a duration of τ. Subsequently, some of the companies saw recovery at time $t - \tau$, return to the susceptible class according to the rate $\delta e^{-\gamma_L \tau} I(t - \tau)$. Specifically, the immunity period τ represents the model's time lag. Additional discussion on the temporal delay representation of financial immunity is supplied in the next Sect. 5 inside the framework of risk contagion among Italian small and medium-sized enterprises.

Supposing that some players leave the economic sector of interest indefinitely is another assumption. As already pointed out, it makes sense to assume that the elimination rate is based on the risk level of the compartments. More specifically, we denote the death rates of low-risk and high-risk companies by γ_L and γ_H, respectively, and notice that it is reasonable to assume, from a financial standpoint, that $0 < \gamma_L \leq \gamma_H < 1$. It means that the removal rate γ_L related to susceptible and recovered players is smaller than the removal rate γ_H linked to agents who are infected. From this perspective, the percentage of susceptible and recovered companies leaving the market is modeled by the terms $\gamma_L S(t)$ and $\gamma_L R(t)$. Conversely, $\gamma_H I(t)$ simulates the percentage of infected agents that exit indefinitely the economic sector.

On the other hand, susceptible compartment grows over time in accordance with the number of new companies that enter the market at a specific growth rate $b > 0$.

Under all previous assumptions, the following SIR dynamics with time delay model the transmission of risk throughout the economic system of interest:

$$\frac{dS(t)}{dt} = b - \gamma_L S(t) - a\, S(t)\, I(t) + \delta e^{-\gamma_L \tau} I(t - \tau), \tag{1}$$

$$\frac{dI(t)}{dt} = a\, S(t)\, I(t) - (\gamma_H + \delta) I(t), \tag{2}$$

$$\frac{dR(t)}{dt} = \delta I(t) - \gamma_L R(t) - \delta e^{-\gamma_L \tau} I(t - \tau). \tag{3}$$

We remark that Eq. (3) can be omitted without loss of generality, as the first two Eqs. (1) and (2) do not depend on the third one which can be integrated once the infected dynamics is known so that

$$R(t) = \delta \int_{t-\tau}^{t} e^{\gamma_L (s-t)} I(s)\, ds,$$

when the initial condition

$$R(0) = \delta \int_{-\tau}^{0} e^{\gamma_L s} I(s)\, ds,$$

is prescribed. In this respect, we only focus on system (1)–(2); then we assume that the problem is completed by the following suitable initial conditions:

$$S(0) = S_0 > 0, \tag{4}$$

$$I(s) = I_0(s) \geq 0, \quad \text{for all } s \in [-\tau, 0], \quad \text{with } I_0(0) > 0, \tag{5}$$

where the history function $I_0(\cdot)$ is continuous in the whole time lag interval $[-\tau, 0]$. Under the previous assumptions, it is possible to prove the following result which states the well-posedness of the model in terms of existence and uniqueness of the solution.

Proposition 1 *The differential system* (1)–(2) *completed by* (4)–(5) *has a unique solution. Moreover, it is possible to prove that the solution gets positive values, i.e.* $S(t) > 0$ *and* $I(t) > 0$ *for all* $t \geq 0$.

Proof The well known results of the fundamental theory of functional differential equations (see Hale 1977) assure that the problem admits a unique solution, due to the assumption that the history function $I_0(\cdot)$ belongs to the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+$.

In the next step, we prove that the solution remains positive so that $S(t) > 0$ and $I(t) > 0$ for all $t \geq 0$, as it starts from positive initial conditions. Actually, we suppose that this assertion is false, which implies that $T = \{t \;\; t > 0,\ S(t) \cdot I(t) = 0\} \neq \emptyset$. Therefore we set $t_1 = \min T$, meaning that t_1 is the first time when $S(t_1) \cdot I(t_1) = 0$.

Under this assumption, we firstly suppose that $S(t_1) = 0$: as a consequence, we get $I(t) \geq 0$ for $t \in [-\tau, t_1]$. Thus, by evaluating (1) at $t = t_1$, we have

$$\left.\frac{dS(t)}{dt}\right|_{t=t_1} = b + \delta e^{-\gamma_L \tau} I(t_1 - \tau) > 0. \tag{6}$$

Since $S(0) = S_0 > 0$, in correspondence with $S(t_1) = 0$ we must get $\frac{dS(t)}{dt}|_{t=t_1} \leq 0$, which is in contradiction with the previous inequality (6).

Therefore, by discarding the possibility that $S(t_1)0$, we suppose that $I(t_1) = 0$. This condition yields $S(t) \geq 0$ for $t \in [0, t_1]$. In this respect, as $S(t)$ is a differentiable function, then it is continuous and bounded over $[0, t_1]$; therefore, it is possible to define

$$C = \min_{0 \leq t \leq t_1} \{aS(t) - (\gamma + \delta)\},$$

so that we obtain $\frac{dI(t)}{dt} \geq CI(t)$ for $t \in [0, t_1]$. By integration in time, it follows that $I(t_1) \geq I_0(0)\, e^{Ct_1} > 0$, which is in contradiction with the assumption $I(t_1) = 0$.

As a result, the set T is empty, meaning that the product $S(t) \cdot I(t)$ never nullifies. Then $S(t)$ and $I(t)$ remain positive for any time t. □

In this framework, we account for the total density $N(t) = S(t) + I(t) + R(t)$ for all $t \geq 0$ and suppose that the system is normalized such that the initial total density corresponds to $N(0) = 1$. The dynamics of $N(T)$ is described by the following equation

$$\frac{dN}{dt} = b - \gamma_L S(t) - \gamma_H I(t) - \gamma_L R(t);$$

thus, condition $\gamma_L \leq \gamma_H$ yields

$$\frac{dN}{dt} \leq b - \gamma_L N(t).$$

By integrating in time, it follows that

$$N(t) \leq N(0)e^{-\gamma_L t} + \frac{b}{\gamma_L}(1 - e^{-\gamma_L t}) = e^{-\gamma_L t} + \frac{b}{\gamma_L}(1 - e^{-\gamma_L t}). \tag{7}$$

We set

$$M = \max\{1; b/\gamma_L\},$$

and notice that the right-hand side in (7) can be bounded according to the following relationship

$$e^{-\gamma_L t} + \frac{b}{\gamma_L}(1 - e^{-\gamma_L t}) \leq M e^{-\gamma_L t} + M(1 - e^{-\gamma_L t}) = M,$$

in order to get $N(t) \leq M$ for all $t \geq 0$. Due to the fact that all the variables get non-negative values, the previous inequality on $N(t)$ yields that also $S(t)$, $I(t)$ and $R(t)$ are upper bounded by M; in other words, at any time $t \geq 0$, it holds that

$$0 \leq S(t) \leq M, \qquad 0 \leq I(t) \leq M, \qquad 0 < R(t) \leq M. \tag{8}$$

2.1 The existence of equilibrium points

Any equilibrium point $E^* = (S^*, I^*)$ characterizing the dynamics at the steady state satisfies the following system of equations

$$\begin{aligned} &b - \gamma_L S^* - a\, S^*\, I^* + \delta e^{-\gamma_L \tau} I^* = 0, \\ &a\, S^*\, I^* - (\gamma_H + \delta) I^* = 0. \end{aligned}$$

Therefore, it is not so difficult to verify that model (1)–(2) admits the risk-free equilibrium $E_0^* = \big(b/\gamma_L, 0\big)$ and one more non-zero steady state $E_\tau^* = \big(S_\tau^*, I_\tau^*\big)$ with

$$S_\tau^* = \frac{\gamma_H + \delta}{a}, \qquad I_\tau^* = \frac{\gamma_L(\gamma_H + \delta)}{a(\gamma_H + \delta - \delta e^{-\gamma_L \tau})}(\rho_0 - 1),$$

where ρ_0 is the basic reproduction number defined by

$$\rho_0 = \frac{ba}{\gamma_L(\gamma_H + \delta)}.$$

The non-trivial equilibrium E_τ^* represents an endemic or not-free-risk steady state and is feasible under the assumption that $\rho_0 > 1$.

3 Risk-free steady state stability

We are interested in determining sufficient conditions which ensure a long-term level of risk-free activity in the economic system. In this regard, we carry out a thorough stability study of the equilibrium point E_0^*.

Specifically, the results in the next Proposition 2 explain how system (1)–(2) behaves close to this stationary state.

Proposition 2 *The following statements hold*:

- If $\rho_0 < 1$, then the risk-free equilibrium E_0^* is locally asymptotically stable and no other equilibrium is feasible;
- In the opposite case when $\rho_0 > 1$, then E_0^* is unstable.

Proof We linearize system (1)–(2) near the steady state E_0^* and focus on the characteristic equation with respect to λ defined as

$$\det(J_0(E_0^*) + e^{-\lambda\tau} J_\tau(E_0^*) - \lambda J) = 0, \tag{9}$$

where J is the identity matrix and

$$J_0(E_0^*) = \begin{pmatrix} -\gamma_L & (\gamma_H + \delta)\rho_0 \\ 0 & (\gamma_H + \delta)(\rho_0 - 1) \end{pmatrix}, \qquad J_\tau(E_0^*) = \begin{pmatrix} 0 & \delta e^{-\gamma_L \tau} \\ 0 & 0 \end{pmatrix}.$$

Therefore, (9) is equivalent to the following equation

$$(\lambda + \gamma_L)(\lambda - (\gamma_H + \delta)(\rho_0 - 1)) = 0;$$

which is solved by real eigenvalues corresponding to $\lambda_1 = -\gamma_L$ and $\lambda_2 = (\gamma_H + \delta)(\rho_0 - 1)$.

Under the assumption that $\rho_0 < 1$, we have both $\lambda_1 < 0$ and $\lambda_2 < 0$. It follows that E_0^* is locally asymptotically stable; moreover, the equilibrium point E_τ^* is not feasible (i.e. $I_\tau^* < 0$).

In the opposite case when $\rho_0 > 1$, we get $\lambda_1 < 0$ and $\lambda_2 > 0$; this conditions implies that E_0^* is unstable. Then, the proof is completed. □

For the sake of completeness, we remark that the assumption $\rho_0 < 1$ in Proposition 2 can be strengthened to state a sufficient condition for assuring the global asymptotic stability of E_0^*, according to the next result.

Proposition 3 *Under the assumption that*

$$\rho_0 < \min\{1; b/\gamma_L\}, \tag{10}$$

the risk-free equilibrium E_0^* is globally asymptotically stable.

Proof We perform the following transformation

$$u_1(t) = S(t) - \frac{b}{\gamma_L}, \qquad u_2(t) = I(t),$$

so that model (1)–(2) is equivalent to the system

$$\frac{du_1(t)}{dt} = -\gamma_L u_1(t) - a\,S(t)\,u_2(t) + \delta e^{-\gamma_L \tau}\,u_2(t-\tau), \tag{11}$$

$$\frac{du_2(t)}{dt} = \big(a\,S(t) - (\gamma_H + \delta)\big)u_2(t). \tag{12}$$

As $0 \le S(t) \le M$ in (8), then the right-hand side in (12) is upper bounded according to

$$a\,S(t) - (\gamma_H + \delta) \le \bar{C}, \tag{13}$$

where we set $\bar{C} = aM - (\gamma_H + \delta)$. It is possible to verify that $\bar{C} < 0$; indeed, two different cases are distinguished:

- If $b/\gamma_L \le 1$, then we get $M = 1$ together with

$$\bar{C} = (\gamma_H + \delta)\frac{\gamma_L}{b}\left(\rho_0 - \frac{b}{\gamma_L}\right) < 0,$$

  which yields $\bar{C} < 0$, due to condition (10);
- In the opposite case when $b/\gamma_L > 1$, then we have $M = b/\gamma_L$ together with

$$\bar{C} = (\gamma_H + \delta)\big(\rho_0 - 1\big),$$

  which yields $\bar{C} < 0$, due to condition (10) again.

We employ (13) in Eq. (12) and obtain

$$\frac{du_2(t)}{dt} \le \bar{C}\,u_2(t),$$

which holds $u_2(t) \le I_0(0)\,e^{\bar{C}t}$. It means that $u_2(t)$ is bounded from above by an exponentially decaying function, since $\bar{C} < 0$; then $u_2(t) = I(t) \to 0$ as $t \to +\infty$.

In addition, the previous bounds on $u_2(t)$ yield $u_2(t) \ge 0$ and $u_2(t-\tau) \le I_0(0)\,e^{\bar{C}(t-\tau)}$, which can be plugged into the right-hand side of Eq. (11) in order to have

$$\frac{du_1(t)}{dt} \le -\gamma_L u_1(t) + \delta e^{-\gamma_L \tau}\,I_0(0)\,e^{\bar{C}(t-\tau)}.$$

This condition can be integrated by distinguishing the following cases:

- If $\gamma_L + \bar{C} \ne 0$, by integration we obtain

$$u_1(t) \le u_1(0)e^{-\gamma_L t} + \delta e^{-(\gamma_L + \bar{C})\tau}\,I_0(0)\,\frac{e^{\bar{C}t} - e^{-\gamma_L t}}{\gamma_L + \bar{C}};$$

- In the opposite case when $\gamma_L = -\bar{C}$, by integration we get

$$u_1(t) \le (u_1(0) + \delta\,I_0(0)t)e^{-\gamma_L t}.$$

In both previous cases, $u_1(t)$ is bounded from above by the an exponentially decaying function; thus, as $t \to +\infty$, we get $u_1(t) \to 0$. As a conclusion, Proposition 3 is proved since we get $S(t) \to b/\gamma_L$ and $I(t) \to 0$ when $t \to +\infty$. □

4 Not-free-risk equilibrium stability

In this Section we assume that $\rho_0 > 1$ which means that E^*_τ is feasible. Then we deal with the endemic equilibrium's stability.

At first, we start from proving the following result which provides a sufficient condition for local stability. Precisely, it discusses the system behaviour at the not-free risk steady state by applying matrix theory for characteristic values.

Proposition 4 *We suppose that $\rho_0 > 1$, then E^*_τ is a feasible equilibrium. Let $p(\cdot)$ be the polynomial function defined by*

$$p(\omega) = \omega^2 - 2(\gamma_H + \delta - \gamma_L)\omega + \gamma_L^2,$$

for any $\omega \in \mathbb{R}$. In correspondence with the fixed time delay $\tau \geq 0$, consider the value p_0 of $p(\cdot)$ at $\omega = aI^*_\tau$ and assume that it is non-negative, i.e.

$$p_0 := p(aI^*_\tau) \geq 0. \tag{14}$$

Then the not-free-risk equilibrium E^*_τ is locally asymptotically stable in correspondence with $\tau \geq 0$.

Proof We linearize system (1)–(2) near the steady state E^*_τ and focus on the characteristic equation with respect to λ defined as

$$\det(J_0(E^*_\tau) + e^{-\lambda\tau} J_\tau(E^*_\tau) - \lambda J) = 0, \tag{15}$$

where J is the identity matrix and

$$J_0(E^*_\tau) = \begin{pmatrix} -\gamma_L - aI^*_\tau & -(\gamma_H + \delta) \\ aI^*_\tau & 0 \end{pmatrix}, \qquad J_\tau(E^*_\tau) = \begin{pmatrix} 0 & \delta e^{-\gamma_L \tau} \\ 0 & 0 \end{pmatrix}.$$

We notice that (15) is equivalent to the following equation

$$\lambda^2 + (\gamma_L + aI^*_\tau)\lambda + (\gamma_H + \delta)aI^*_\tau - aI^*_\tau \delta e^{-(\gamma_L+\lambda)\tau} = 0. \tag{16}$$

When $\tau = 0$, Eq. (16) becomes

$$\lambda^2 + (\gamma_L + aI^*_\tau)\lambda + \gamma_H aI^*_\tau = 0;$$

it has positive coefficients and admits roots with negative real parts. Hence, condition $\rho_0 > 1$ implies that the endemic equilibrium E^*_τ is locally stable when $\tau = 0$.

On the other hand, in the case when $\tau > 0$, we suppose that $\lambda = iv$ $(v > 0)$ is a solution of Eq. (16). Then, by substitution, we obtain

$$-v^2 + (\gamma_L + aI^*_\tau)iv + (\gamma_H + \delta)aI^*_\tau - aI^*_\tau \delta e^{-\gamma_L \tau} e^{-iv\tau} = 0.$$

After separating real and imaginary parts, we get the following equations

$$-\nu^2 + (\gamma_H + \delta)aI^*_\tau = aI^*_\tau \delta e^{-\gamma_L \tau} \cos(\nu\tau),$$
$$(\gamma_L + aI^*_\tau)\nu = -aI^*_\tau \delta e^{-\gamma_L \tau} \sin(\nu\tau),$$

which can be squared and added in order to have

$$\nu^4 + p_0 \nu^2 + (aI^*_\tau)^2 \left(\gamma_H^2 + 2\gamma_H \delta + \delta^2 (1 - e^{-2\gamma_L \tau})\right) = 0, \tag{17}$$

where we have $p_0 := p(aI^*_\tau)$ according to assumption (14). We notice that the sign of coefficient p_0 is crucial: actually, condition (14) implies that Eq. (17) has positive coefficients and cannot be solved. As well-known, it follows that E^*_τ is locally stable. Thus the proof is complete. □

As previously stated, Proposition 4 establishes condition (14) which is crucial in order to assure local stability at the not-free risk steady state. In this respect, the following result essentially serves as a corollary that provides some suitable assumptions about the parameters which guarantee the fulfillment of the preceding condition (14) in correspondence with any value of time delay.

Corollary 5 *We suppose that $\rho_0 > 1$; thus E^*_τ is a feasible equilibrium. The following statements hold*:

S_1: If γ_L is sufficiently large such that $\gamma_L \geq (\gamma_H + \delta)/2$, then E^*_τ is locally asymptotically stable for any delay $\tau \geq 0$;

S_2: In the opposite case when $\gamma_L < (\gamma_H + \delta)/2$, we set $\omega_2 = \gamma_H + \delta - \gamma_L + \sqrt{(\gamma_H + \delta - \gamma_L)^2 - \gamma_L^2}$ and notice that the strengthened condition

$$\rho_0 \geq 1 + \frac{\omega_2}{\gamma_L}, \tag{18}$$

assures that the not-free-risk equilibrium is locally asymptotically stable for any delay $\tau \geq 0$.

Proof We are going to prove that the inequality in (14) is verified under the assumptions in the previous statements S_1 and S_2. With this aim, we notice that the discriminant of $p(\cdot)$ is defined as

$$\Delta = 4(\gamma_H + \delta)(\gamma_H + \delta - 2\gamma_L).$$

Concerning statement S_1, if $\gamma_L \geq (\gamma_H + \delta)/2$, then $\Delta \leq 0$ and $p(\omega) \geq 0$ for any value $\omega \in \mathbb{R}$. This yields condition (14) is satisfied in correspondence with any value of time delay $\tau \geq 0$. Therefore, due to Proposition 4, E^*_τ is locally asymptotically stable for any τ.

On the other hand, we focus on statement S_2 and assume that $\gamma_L < (\gamma_H + \delta)/2$. Then, we have $\Delta > 0$ and $p(\omega) \geq 0$ for any $\omega \in (-\infty, \omega_1] \cup [\omega_2, +\infty)$ with

$$\omega_1 = \gamma_H + \delta - \gamma_L - \sqrt{(\gamma_H + \delta - \gamma_L)^2 - \gamma_L^2},$$

and

$$\omega_2 = \gamma_H + \delta - \gamma_L + \sqrt{(\gamma_H + \delta - \gamma_L)^2 - \gamma_L^2}.$$

For the sake of clarity, the previous parameter ω_2 corresponds to the one already defined in statement S_2. We also suppose that the further condition (18) holds; then, it is not so difficult to verify that

$$\omega_2 \le \gamma_L\left(\rho_0 - 1\right) < \gamma_L\left(\rho_0 - 1\right)\frac{\gamma_H + \delta}{\gamma_H + \delta - \delta e^{-\gamma_L \tau}} = aI_\tau^*.$$

It follows that $p_0 = p(aI_\tau^*) > 0$. Again, this implies condition (14) is satisfied in correspondence with any value of time delay $\tau \ge 0$. Due to Proposition 4, it follows that E_τ^* is locally asymptotically stable for any τ. The proof is completed. □

The analysis is completed by the next result dealing with some sufficient conditions which guarantee global stability at the endemic steady state.

Proposition 6 *Under the assumption that $\rho_0 > 1$, E_τ^* is a feasible equilibrium. We set*

$$C_1 = \frac{\delta^2(\gamma_L + \gamma_H + \delta)}{2\gamma_L(\gamma_H + \delta)}, \qquad C_2 = \frac{\gamma_H + \delta}{2},$$

and again

$$C = aM - (\gamma_H + \delta),$$

as in the previous Sect. 3. We suppose that

$$C_2 > C_1.$$

The following statements hold:

S_3: If ρ_0 is sufficiently large such that

$$\rho_0 \ge 1 + \frac{\gamma_L + \gamma_H + \delta}{\gamma_L} \cdot \frac{C}{C_2 - C_1}, \tag{19}$$

then the steady state E_τ^* is globally asymptotically stable for any delay $\tau \ge 0$;

S_4: In the other case when

$$1 + \frac{\gamma_H(\gamma_L + \gamma_H + \delta)}{\gamma_L(\gamma_H + \delta)} \cdot \frac{C}{C_2 - C_1} \le \rho_0 < 1 + \frac{\gamma_L + \gamma_H + \delta}{\gamma_L} \cdot \frac{C}{C_2 - C_1}, \tag{20}$$

equilibrium E_τ^* is globally asymptotically stable for any delay $\tau \le \tau_0$, where the threshold τ_0 is defined as

$$\tau_0 = \log\left(C_3\right)/\delta,$$

with

$$C_3 = \frac{\delta}{(\gamma_H + \delta)\Big(1 - \frac{C_2 - C_1}{C} \cdot \frac{\gamma_L}{\gamma_L + \gamma_H + \delta}(\rho_0 - 1)\Big)}.$$

Proof We center system (1)–(2) at E^*_τ by introducing new variables defined by

$$u_1(t) = S(t) - S^*_\tau, \qquad u_2(t) = I(t) - I^*_\tau;$$

therefore, the problem can be rewritten in the following form

$$\begin{aligned}\frac{du_1(t)}{dt} &= -\gamma_L u_1(t) - aS(t)(u_2(t) + I^*_\tau) + \delta e^{-\gamma_L \tau} u_2(t-\tau) + (\gamma_H + \delta)I^*_\tau,\\ \frac{du_2(t)}{dt} &= \big(aS(t) - (\gamma_H + \delta)\big)(u_2(t) + I^*_\tau).\end{aligned}$$

We are going to show that the trivial solution $(u_1 = 0, u_2 = 0)$ of this system is globally asymptotically stable, then it will immediately follow that the endemic equilibrium E^*_τ is globally asymptotically stable for model (1)–(2). Then we introduce the following functional

$$V(u(t)) = \frac{1}{2}(u_1(t) + u_2(t))^2 + \frac{1}{2}\bar{\varpi} \cdot u_2^2(t),$$

where $u(t) = (u_1(t), u_2(t))$ and

$$\bar{\varpi} = \frac{\gamma_L + \gamma_H + \delta}{aI^*_\tau} > 0. \tag{21}$$

The derivative of V is

$$\begin{aligned}\frac{dV}{dt} =& (u_1(t) + u_2(t))\big(-\gamma_L u_1(t) - (\gamma_H + \delta)u_2(t) + \delta e^{-\gamma_L \tau} u_2(t-\tau)\big)\\ &+ \bar{\varpi}\big(aS(t) - (\gamma_H + \delta)\big)u_2^2(t) + \bar{\varpi}\, aI^*_\tau u_1(t)u_2(t).\end{aligned}$$

Due to (21), the previous relationship is equivalent to

$$\begin{aligned}\frac{dV}{dt} =& -\gamma_L u_1^2(t) - (\gamma_H + \delta)u_2^2(t) + (u_1(t) + u_2(t)) \cdot \delta e^{-\gamma_L \tau} u_2(t-\tau)\\ &+ \bar{\varpi}\big(aS(t) - (\gamma_H + \delta)\big)u_2^2(t).\end{aligned}$$

As $aS(t) \le aM - (\gamma_H + \delta) = C$ and $e^{-\gamma_L \tau} \le 1$, then the previous derivative is bounded from above according to the following estimate

$$\frac{dV}{dt} \le -\gamma_L u_1^2(t) - (\gamma_H + \delta - \bar{\varpi} C)u_2^2(t) + (u_1(t) + u_2(t)) \cdot \delta u_2(t-\tau). \tag{22}$$

By applying Cauchy-Schwartz inequality to $(u_1(t) + u_2(t)) \cdot \delta u_2(t-\tau)$, we get

$$(u_1(t) + u_2(t)) \cdot \delta u_2(t-\tau) \le \frac{\gamma_L}{2}u_1^2(t) + \frac{\gamma_H + \delta}{2}u_2^2(t) + \frac{\delta^2(\gamma_H + \delta + \gamma_L)}{2\gamma_L(\gamma_H + \delta)}u_2^2(t-\tau).$$

The previous bound is applied on (22) in order to obtain

$$\frac{dV}{dt} \le -\frac{\gamma_L}{2}u_1^2(t) + C_1 u_2^2(t-\tau) - \left(C_2 - \bar{\omega}C\right)u_2^2(t).$$

Then we consider Lyapunov functional of the form

$$U(u(t)) = V(u(t)) + C_1 \int_{t-\tau}^{t} u_2^2(\theta)d\theta,$$

hence

$$\begin{aligned}\frac{dU}{dt} =& \frac{dV}{dt} + C_1 u_2^2(t) - C_1 u_2^2(t-\tau) \\ \le& -\frac{\gamma_L}{2}u_1^2(t) - \left(C_2 - \bar{\omega}C - C_1\right)u_2^2(t).\end{aligned}$$

Starting from assumption $C_2 > C_1$, we are interested to get $C_2 - \bar{\omega}C - C_1 > 0$ that is equivalent to $\bar{\omega} < (C_2 - C_1)/C$ or

$$(\gamma_H + \delta)\left(1 - \frac{\gamma_L}{\gamma_L + \gamma_H + \delta} \cdot \frac{C_2 - C_1}{C}(\rho_0 - 1)\right) < \delta e^{-\gamma_L \tau}, \tag{23}$$

where $\bar{\omega}$ is replaced by its value according to (21). In this respect, we distinguish two different situations related to statements S_3 and S_4. Concerning proposition S_3, we notice that assumption (19) assures that the left-hand side in (23) is negative, therefore condition $C_2 - \bar{\omega}C - C_1 > 0$ is satisfied without any restriction on delay τ.

On the other hand, assumption (20) in statement S_4 implies that the left-hand side in (23) gets value between 0 and δ so that $\tau_0 \ge 0$ and condition $C_2 - \bar{\omega}C - C_1 > 0$ holds for any $\tau \le \tau_0$.

As a conclusion, the assumptions in both statements S_3 and S_4 assure that U is negative definite. A direct application of the Lyapunov–LaSalle type theorem (see Theorem 2.5.3 in Kuang 1993, p. 30) implies that both $u_1(t) \to 0$ and $u_2(t) \to 0$ as $t \to +\infty$. Then equilibrium E_τ^* is globally asymptotically stable in the case when the assumptions in S_3 and S_4 hold, thus the proof is completed. □

5 Numerical simulations

We exploit data from AIDA - BUREAU VAN DIJK, which is a database containing extensive details on Italian enterprises, and we provide a few numerical simulations. Specifically, we address two scenarios: the first one concerns the automotive sector across the whole Italian territory, and the second scenario relates to the food sector in the Emilia Romagna area of Italy.

With the aim of identifying companies in the automotive sector's purview, in the AIDA database we select all companies that have been classified under NACE rev. 2 codes 2910, 292, 2931, 2932, during the years 2013 through 2021. In total we look at 3520 companies which have been active in Italy during the entire time period, encompassing a diverse spectrum of conditions, including companies in good financial health, companies facing financial challenges, companies undergoing resolution processes, or companies encountering bankruptcy proceedings. This selection process yields an inherently unbalanced sample. It's crucial to note that our approach allows for annual variations in the composition of companies due to factors such as new entries, closures, and financial distress. We conduct a comprehensive analysis of all available fundamental and balance sheet data for these companies.

Regarding the food industry, we use a similar methodology, identifying companies in this sector using NACE rev. 10 classifications for the years 2012–2021 based on information retrieved from the AIDA database. In this case, we focus on 2188 companies located in the Emilia Romagna area, encompassing a broad range of circumstances again, such as financially stable companies, companies facing difficulties, companies going through resolution procedures, or companies which have been involved in bankruptcy proceedings. Again, this methodology yields an unbalanced sample, and we thoroughly examine all accessible fundamental and balance sheet data.

We adopt a straightforward approach to classify the conditions of these companies. Actually, companies that have incurred a financial loss in the preceding year are labeled as "infected" while companies that have reported losses in the prior year but manage to generate profits in the current year are classified as "recovered". This classification serves as the foundation for our parameter creation for the applications. We decide for this level of simplicity since, even with the use of various techniques such as Altman scoring criteria and others, the proportion of infected and healthy companies does not change.

On one hand, the AIDA data are employed in order to estimate the parameters a, b, δ, γ_L and γ_H which are involved in the dynamic model, since our main focus is understanding its long-term evolution. In this sense AIDA data are considered as in-sample data, as they are used during the training or calibration of the model. We observe that the evaluation of the model's robustness and generalizability may benefit from the integration of out-of-sample data. Evaluating the model's ability to forecast beyond the initial dataset could be greatly enhanced by testing it using such data. Nevertheless, this issue is outside the focus of the work, but adding more historical or external observations to our dataset might be a fruitful path for future investigation, enhancing the significance and generalizability of our results.

On the other hand, we observe that all the parameters of the model are determined by calibrating data with the exception of time delay τ, which is not measured. In this regard, we would note that the idea of "financial immunity" is mathematically modeled by this time delay, and this serves as a tool for understanding the observed behavior of companies as they move from one state of financial difficulty to another. Our underlying assumption is that following an initial or repeated financial distress situation, various stakeholders (i.e. shareholders, financial intermediaries, and other relevant parties) may take actions to temporarily prevent a recurrence of financial distress.

For instance, following a period of financial distress, shareholders tend to exercise more vigilant monitoring activities, and regulatory authorities may increase their oversight, especially if the company is publicly traded. In the case of small and medium-sized enterprises (SMEs), as is often seen in the Italian context, banks utilize mechanisms like the "Centrale dei rischi" of the Bank of Italy to continuously assess the credit risk of these companies, functioning as a sort of financial watchdog. Furthermore, in the aftermath of a financial distress period, companies typically formulate contingency and rescue plans. During this phase, the company builds financial robustness, akin to developing immunity, which serves as a safeguard against future financial troubles. Under a mathematical perspective, time delay τ in this framework denotes the temporal period when acting to control financial trouble from happening again.

As a further remark, we notice that the parameters involved in the dynamics are evaluated by exploiting data spanning around a decade. Even though the data's time series are not very long, both industries have shown value. Indeed, the results provided in the following Sects. 5.1

and 5.2 allow us to deal with distinct scenarios in the long run, as our main topic of study is the asymptotic behavior of financial risk spread at the steady state.

Finally, we would like to remark that model (1)–(2) does not admit solution in closed form, due to its nonlinearity. Then, we approximate the solution in MATLAB environment.

In this regard, for each economic sectors we denote by N_0 the number of companies in the data sample at $t = 0$. Initial conditions S_0 and $I_0(\cdot)$ are estimated by normalizing with respect to N_0 so that the total density $N(\cdot)$ starts from the unitary value, i.e. $N(0) = 1$; in this way, upper bounds in (8) are satisfied at any time. Furthermore, the density of infected companies for each year of the temporal period of the sample are used to get the nodal values for history function $I_0(\cdot)$. Then, the MATLAB built-in function `polyfit` is employed for evaluating a polynomial of a given degree as best fit (in a least-squares sense) for the nodal values of $I_0(t)$ for $t \leq 0$.

5.1 Automotive sector in Italy

Over the past few years, this economic sector has experienced numerous changes on a global scale, which are then seen locally. The goal of electric cars for European nations (and other G-20 automakers) has an impact on both the production process and environmental emission regulation. In addition, the importance of the global supply chain has increased, particularly since COVID-19, which caused the alleged chip shortage. In this regard, a confluence of various events created the global chip crisis, with the COVID-19 pandemic's snowball effect acting as the primary cause of growing shortages. Another factor that plays a role is the fact that demand is outpacing production capability. Other factors have also been implicated in the drought in Taiwan in 2021 and the trade dispute between China and the United States. Big international automakers' M&A decisions are another trait of this industry (i.e. Fiat and Chrysler create FCA; after FCA and Peugeot create Stellantis). The previous hints about the changes for this sector are the starting point for our simulation related to the Italian sector and its supply chain at the level of the companies.

Concerning our simulation in terms of descriptive statistics, we sketch a summary of the Return On Equity (ROE) panel. We exploit the numerator of this indicator as a proxy for financial distress, as this approach offers a simpler yet effective alternative to more complex methods, such as the Altman Score. Crucially we find that, in contrast to these more complex methods, using ROE does not substantially change the outcomes. This choice allows us to maintain analytical rigor while ensuring the model remains accessible and understandable.

The ROE descriptive data for the sample of companies in the automotive industry are shown in Table 1. The data show significant trends, such as a notable decline in industry profitability in 2020 brought on by the extensive effects of COVID-19. In addition, the Table shows the impact of the economic situation in Italy in 2012, which similarly had a negative effect on the ROE of the sector. These events highlight the industry's vulnerability to macroeconomic shocks and the importance of understanding how external factors influence financial performance over time.

Table 1 Mean return on equity %—automotive industries

	2013	2014	2015	2016	2017	2018	2019	2020	2021
ROE%	0.95	4.27	8.55	9.58	11.71	10.97	10.70	6.55	11.87

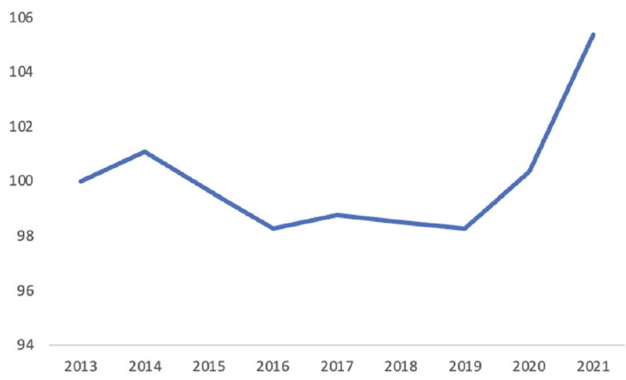

Fig. 2 Index of companies operating in Automotive Sector (2013 = 100)

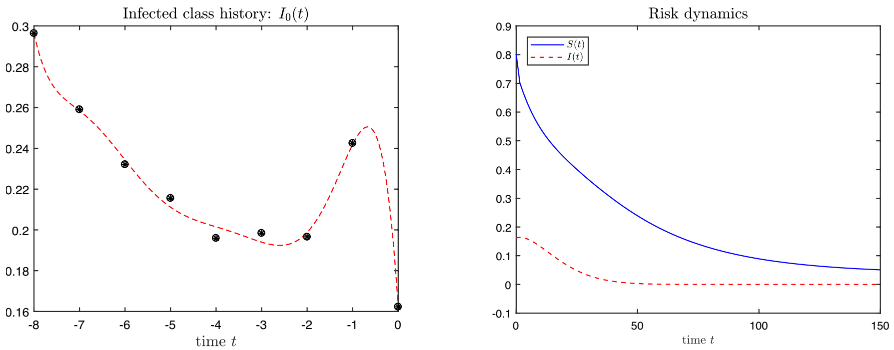

Fig. 3 Left: plot of history function $I_0(t)$ in the whole sample period $[-8, 0]$. Right: dynamics related to automotive sector; the solid blue line describes susceptible dynamics, the dashed red line corresponds with infected dynamics. (Colour figure online)

In Fig. 2, we show the indices representing the number of companies that operated during the years analyzed in the sample. We choose to report indices instead of absolute numbers because our simulations rely on the density function. the use of indices enables us to maintain consistency with the analytical framework of our simulations, in order to get a more accurate representation of the distribution and behavior of companies within the sample over time.

As already mentioned, the data sample is collected over the time horizon between 2013 and 2021 such that the whole sample period is $[-8, 0]$, whose length corresponds with 8 years. The nodal values for infected density are estimated as the entries of the vector (265, 396, 321, 324, 320, 352, 379, 423, 484) which is normalized with respect to the initial number of companies $N_0 = 1633$ operating at $t = 0$. We apply the built-in function `polyfit` by setting the best fit polynomial degree at $n = 7$ in order to estimate history function $I_0(t)$ for $t \in [-8, 0]$, which is provided on the left of Fig. 3. The nodal values of infected density are represented by black bullets and the red dashed line is the plot of $I_0(\cdot)$.

By exploiting the data from AIDA database, we may estimate that risk evolves according to the values of the parameters given by $a = 0.2760$, $b = 0.01061$, $\delta = 0.1527$, $\gamma_L = 0.0277$ and $\gamma_H = 0.0426$. In this case it is possible to verify that $\rho_0 < 1$, thus risk-free equilibrium

Table 2 Mean return on equity %—food industries

	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021
ROE%	3.13	4.5	5.92	7.5	8.6	9.07	9.32	9.08	7.04	7.96

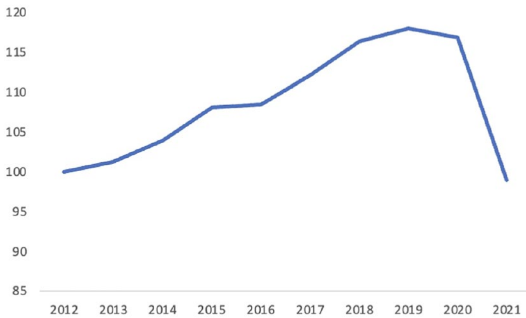

Fig. 4 Index of companies operating in the Food Sector in Emilia Romagna (2012 = 100)

is locally asymptotically stable for any time delay τ and not-free-risk steady state is not feasible. We notice that time delay τ is not measured; therefore, we set $\tau = 1$ in order to perform a numerical simulation whose result is shown on the right of Fig. 3. It is evident that both densities $S(t)$ and $I(t)$ decrease in time over the whole temporal horizon and they converge towards the risk-free equilibrium as $t \to +\infty$. It means that risk disappears from the automotive economic sector in the long run. As expected, risk contagion dynamics has similar behaviour at the steady state when the time value for τ is set different from 1.

For the sake of fairness, we should note that the simulation depends in the short run on the degree n of the best fit polynomial $I_0(\cdot)$ which is chosen equal to the maximum possible value 7 for the considered data. Actually, due to the fact that the oscillations of $I_0(\cdot)$ dampen as n decreases, the dynamics of the SIR model depends in the transient on both the degree n and the length of the lag period. Instead, in the framework of the asymptotic analysis we focus on for predicting whether the risk remains at equilibrium, the degree n has no bearing on the steady-state dynamic since it depends on the model's parameters that are included in the basic reproduction number which does not exceed threshold 1.

5.2 Food sector in Emilia Romagna Italian region

This case study has been already simulated in Aliano et al. (2023) by a different SIR model. A sample of companies operating in the food sector of Emilia Romagna Italian region is considered. This region is located in the northern part of Italy and its food sector plays a key economic role in the entire territory of northern Italy.[2]

The descriptive statistics for the ROE of companies within the Food industry sample in Emilia Romagna are shown in Table 2.

The data reveal significant trends that are closely correlated with the region's economic growth, reflecting the dynamic relationship between local economic conditions and

[2] https://www.bancaditalia.it/pubblicazioni/economie-regionali/?dotcache=refresh website consulted in December 2022.

industry performance. Notably, the table also highlights the adverse effects of the 2012 economic situation in Italy, which negatively impacted the ROE of the sector. These observations emphasize the Food industry's susceptibility to macroeconomic shocks and reinforce the importance of analyzing how external factors can shape financial outcomes over time.

In Fig. 4, the indices representing the number of companies that operated in the Food sector in Emilia Romagna during the years analyzed in the sample are plotted.

The time horizon related to the data sample is set between 2012 and 2021 and its length corresponds with 9 years, therefore the whole sample period is $[-9, 0]$. The entries of the vector (232, 217, 204, 186, 173, 174, 196, 217, 282, 163) are normalized with respect to the initial number of companies $N_0 = 1744$ operating at $t = 0$, in order to estimate the nodal values for infected density. Again, history function $I_0(t)$ is evaluated for any t in the whole time horizon $[-9, 0]$, by employing the built-in function `polyfit` and setting the best fit polynomial degree at $n = 7$. We remark that the same considerations about degree n we have pointed out in the previous Sect. 5.1 also apply to this simulation. The plot of $I_0(\cdot)$ is shown on the left of Fig. 5.

The history function makes it straightforward to think about granular risk. Actually, we notice that the plot of $I_0(\cdot)$ in both Figs. 3 and 5 show the percentage of "infected" companies, which we evaluate as a measure of risk. We adopt a simple classification approach to determine the financial health of these companies. In particular, companies that incurred a financial loss the year before are categorized as "infected", whereas companies that recorded losses the year before but were able to turn a profit the following year are categorized as "recovered". The parameters employed in our model are defined based on this classification. We chose this level of simplicity because the proportion of infected versus healthy companies remains largely unchanged, even when more complex techniques such as Altman scoring criteria are employed. Therefore, the simple approach we apply provides a clear and effective framework for our analysis without compromising the accuracy of our findings.

According to the data, the values of the parameters involved in the risk dynamics are estimated as $a = 0.12$, $b = 0.04$, $\delta = 0.05$, $\gamma_L = 0.0086$ and $\gamma_H = 0.0125$. Moreover, time delay is set at $\tau = 1$ in agreement with the same value considered in the previous simulation model in Sect. 5.1. The resulting dynamics of risk contagion is numerically simulated

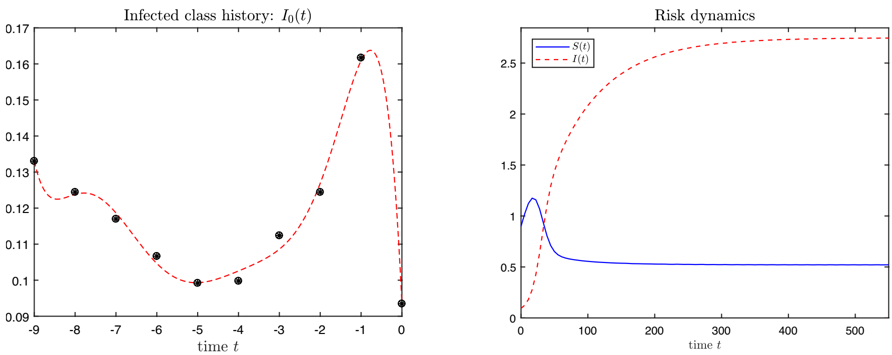

Fig. 5 Left: plot of history function $I_0(t)$ in the whole sample period $[-9, 0]$. Right: dynamics related to food sector; the solid blue line describes susceptible dynamics, the dashed red line corresponds with infected dynamics. (Colour figure online)

and plotted on the right of Fig. 5. It is evident that not-free-risk equilibrium attracts the trajectory of SIR solution: it means that risk infection remains in the economy in the long run. It is due to the fact that condition (14) is satisfied so that endemic steady state is locally asymptotically stable. Moreover, the infected density exceeds the susceptible one at stationary state.

5.3 Different behaviours concerning the economic sectors

The differences between the food and automotive industries provide interesting insights into the dynamics of risk and financial health within different industrial ecosystems. In the food industry, there is a lower infection rate (i.e., proportion of companies experiencing financial distress) than in the automotive industry, but there is also a lower recovery rate. This contrast shows that, in comparison to their counterparts in the automotive industry, fewer companies in the food sector may have initial financial difficulties, and those that do have a lower chance of recovering.

Furthermore, the higher default rate among healthy companies in the food sector compared to the manufacturing sector suggests that this sector is characterized by a greater underlying vulnerability. This may imply that, even in case of financial stability, food companies face more unpredictable or volatile conditions that can push them into distress more easily than manufacturing companies, especially those in the automotive sector.

In this framework, we remark that one of the most crucial differences is the higher recovery rate in the automotive sector. This shows that compared to food industries, automotive companies are better able to recover from financial difficulties. The higher recovery rate plays a significant role in determining the sector's overall stability. This dynamic produces a free-risk equilibrium in the automotive industry, where the industry as a whole has a tendency to stabilize easily, absorb shocks, and recover to a healthy state more quickly.

On the other hand, the food sector does not reach a free-risk equilibrium as easily. The industry is unable to stabilize in the same manner as the automotive sector due to a combination of factors including a greater default rate among healthy enterprises and a lower recovery rate. This indicates that the food sector remains more exposed to ongoing risks and financial instability over time.

These remarks, in conjunction with the idea of financial immunity, imply that the recovery rate of companies is a crucial factor in achieving a free-risk equilibrium. While financial immunity, or a company's resilience to economic shocks, is significant, so is a company's capacity to recover from financial distress. The distinctions between the food and automotive sectors show how a sector's ability to recover from financial distress has a big impact on its long-term equilibrium and overall stability. Therefore, fostering a higher recovery rate and enhancing financial immunity may be crucial for achieving greater stability from financial distress and lowering risk across different industries.

6 Concluding discussion

In this paper we deal with the relevant problem of modelling how financial contagion may spread among enterprises. In the framework of Susceptible-Infected-Recovered dynamics, we analyse an original time delay differential system for risk diffusion among companies in a given economic sector, by modelling contagion evolution in terms of credit and financial risks with low and high levels.

Due to the fact that the paradigm of the model we propose and its discussion can be considered as worthy of attention, we investigate SIR dynamics at the long run. Actually, two equilibrium points are determined: the first one is risk-free, while the second equilibrium represents a not-free-risk steady state. In this respect, Lyapunov functional approach is employed to prove global stability for both equilibria. We confirm that the basic reproduction number ρ_0 plays a fundamental role in the stability of the differential system. Actually, we prove the existence of an upper bound for ρ_0 that assures the global asymptotic stability for risk-free equilibrium, which means that risk can be eliminated from economic sector at the long run. On the other hand, we also provide some original sufficient conditions for global asymptotic stability of not-free-risk steady state; in particular, we state the existence of some thresholds for the basic reproduction number and for the immunity period which guarantee risk will endemically continue to exist in the economy at the long run.

The model is applied to two different production sectors regarding two different geographic divisions.

We have specifically thought about the food industry in the Emilia-Romagna region and the automotive industry in Italy, as well as how credit risk can spread within these two industries. The data are taken from the database AIDA, which restricts its coverage to Italian companies exclusively, resulting in the exclusion of other automotive companies situated in European nations. Furthermore, we have not accounted for the activities of Italian companies engaged in providing intermediary products beyond their national borders. In this respect, even though our study presents the limitations associated with our database selection, it can be considered as a starting point for addressing future research endeavors by broadening the database to encompass a more diverse range of companies and exploring the interrelationships between companies operating across international boundaries, such as Italian automotive companies operating in Germany. Nonetheless, the proposed SIR model can be employed to evaluate the risk of contagion in industries and across national and global economies that are not the focus of our simulations in this paper.

In addition to the existing limitations of our study, we also note that the model assumptions do not account for external factors which could impact bankruptcy and the health of companies, such as a pandemic. Furthermore, we have not taken into consideration other potential channels for risk transmission, such as market risk and the possibility of credit constraints imposed by financial institutions. As already pointed out, our analysis can be considered as a starting point for future research in order to account for potential external factors.

As already mentioned, we have analyzed two different producing sectors and their related supply chains in great detail in our simulation, both of which involved Italian enterprises. These sectors approach different equilibria while using the same value τ for the time of financial immunity. On one hand the food sector is characterized by an endemic diffusion of risk; on the other hand, the automotive sector reaches an equilibrium where the risk tends to vanish.

In the case of the automobile industry, our analysis has included businesses from all around Italy. On the other hand, we have concentrated on the Emilia Romagna region as what concerns the food industry. Every one of these supply chains and production sectors is important in the area that has been studied. Different behaviours and equilibria have been found in these two sectors, which suggests that different strategic activities are required. The food industry is peculiar because of its intra-company relationships and special supply chain finance dynamics, which allow credit risk to spread quickly across the participating companies. There are many ways in which policymakers at the regional and national levels might improve this sector's resilience. First, by putting guarantee plans into place, particularly at the regional level, to help businesses that are having financial difficulties and by prolonging the period of time that these measures provide protection against contagion. Second, by encouraging businesses to proactively reduce the danger of financial distress spreading through the implementation of Decentralized Finance (DeFi) technologies. Here, we have two main goals in mind: first, we want to improve the financing costs and accessibility of Small and Medium-sized Enterprises (SMEs) so that they can become more competitive; second, we want to address the common issues that arise from asymmetric information risks and the associated expenses in the context of supply chain finance. On the other hand, the Italian automobile industry demonstrates a stronger financial interdependence among its enterprises, which leads to a more stable credit risk equilibrium. This industry has a smaller credit risk spread than the food industry, especially when it comes to internal financial problems. Policymakers, operating primarily at the national level, should prioritize their efforts in addressing exogenous crises rather than endogenous ones in the automotive sector.

As a conclusion, this work can represent a starting point for our research in different directions. On one hand, we note that adding white noise to the dynamical model and examining the impact of perturbing the estimate of the coefficient that measures the removal rate due to contagion in a stochastic context would be a possible and natural development of this paper.

On the other hand, since the numerical simulations we have proposed are inspired by panel data, we would aim at better highlighting the contagion effect by applying the compartmental approach for risk diffusion in other contexts where financial institutions such as banks can provide more detailed information in terms of both granular data and time series about risk ratios among businesses.

Finally, incorporating the compartmental approach inside the framework of evolutionary game theory would be a further step towards expanding our research and accounting for potential economic agent collaboration aimed at mitigating or controlling the spread of financial risk.

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Declarations

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Authors and Affiliations

Mauro Aliano[1] · Lucianna Cananà[2] · Tiziana Ciano[3] · Stefania Ragni[1] · Massimiliano Ferrara[4]

✉ Massimiliano Ferrara
massimiliano.ferrara@unibocconi.it

Mauro Aliano
mauro.aliano@unife.it

Lucianna Cananà
lucianna.canana@uniba.it

Tiziana Ciano
t.ciano@univda.it

Stefania Ragni
stefania.ragni@unife.it

1 Department of Economics and Management, University of Ferrara, Via Voltapaletto 11, 44121 Ferrara, Italy

2 Ionian Department of Law, Economics and Environment, University Aldo Moro di Bari, Lago Maggiore Angolo Via Ancona, 74121 Taranto, Italy

3 Department of Economics and Political Sciences, University of Valle d'Aosta, Via Monte Solarolo, 11100 Aosta, Italy

4 Department of Management and Technology, ICRIOS - The Invernizzi Centre for Research in Innovation, Organization, Strategy and Entrepreneurship, Bocconi University, Via Sarfatti, 25, 20136 Milan, Italy