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Cultural semiotics for mathematical discourses

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Abstract: Mathematics is often defined as a "universal" or "conventional" language. Yet, things may be not as simple as that. The theoretical lens of the semiosphere, with the related notions of context and spatial dynamics, within which the concept of cultural conflict is defined, provides a new framework for research in mathematics education to consider the cultural aspects of mathematical discourses. It is under this framework that learning awareness occurs, and teaching challenges are no longer conceived as independent of the content taught (or to be taught). It is not a question of nullifying the cultural conflict, but exploiting the concept of asymmetry to make sense of mathematical discourse. Meeting foreign cultures leads to looking at one's own practices. An example drawn from Danish numerals, juxtaposed with a mathematical discourse occurring in a sixth-grade classroom in Italy, delves into the practical application of the framework.

Keywords: mathematical discourses; cultural conflict; asymmetry; half; Danish numerals

1 Introduction

Mathematics is often defined as a "universal" or "conventional" language. Yet, "as has already been noted by mathematics education researchers, things may be not as simple as that" (Kim et al. 2012: 86). The ethnomathematics strand of research, for example, argues that mathematics is language-dependent: "Language and mathematics both … have grown differently in different cultures. And both have been affected by cultural encounters throughout history" (D'Ambrosio 2000). It is not a question of whether the language one speaks limits what one can say, do, and think mathematically, but rather to use the "mathematical world" that each language contains to understand what a resource it can prove to be for learning-teaching mathematics. As proved by Barton,

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> these worlds exist – they are not just rudimentary versions of conventional mathematics, nor are they simple, unformalised mathematics. These worlds represent *systems of meaning* concerned with quantity, relationships, or space [or "QRS-system" for short], and are, in some sense, incommensurable with [the universal and conventional] mathematics [i.e., what is found in mathematics texts or journals]. (Barton 2008: 144)

In order to illustrate the issue, a basic mathematical example can be provided: a reflection on natural numbers and the Danish numeral system. (I invite those interested in further reflections on numeral systems in different languages to refer to Bazzanella 2011).

The current Danish numeral system is a positional decimal system, with the ten digits from *nul* ('zero') to *ni* ('nine'), whose values depend on their respective positions. However, both in the system and in the language, there are traces of other systems such as the vigesimal system. Consider the multiples of ten. While the words *tyve* ('twenty'), *tredive* ('thirty'), and *fyrre* ('forty'), which etymologically consist of words meaning respectively 'two', 'three', and 'four' with *ti* ('ten'), reflect a decimal system, the numerals from fifty to ninety bear traces of the vigesimal system, since *tyve* ('twenty') is included in the formulation of the single numbers: *halvtredsindstyve* ('fifty'), which literally means "half-three-times-twenty"; *tresindstyve* ('sixty'), namely, "three-times-twenty"; *halvfierdsindstyve* ('seventy'), which literally means "half-four-times-twenty"; *firsindstyve* ('eighty'), namely, "four-times-twenty"; and *halvfemsindstyve* ('ninety'), which literally means "half-five-times-twenty." The truncated forms, namely, *halvtreds*, *tres*, *halvfjerds*, *firs*, and *halvfems*, have become the prevalent usage in contemporary contexts. Nevertheless, the elongated variants, though now regarded as antiquated, persist and support in elucidating etymological origins.

Words for the tens with an odd first digit rely on non-integer multipliers that are prefixed to the numeral of the next tens. These multipliers refer to the next integer unit and not to the previous one, as is the case in Italian for instance. In fact, in Danish, the numerals *halvtredje* ('two and a half') literally is 'half-third', or better 'half [unit before] the third [unit]'.

This leads to the fact that the numeral *halvtredsindstyve* ('fifty'), which literally means "half-three-times-twenty" implies the counting of "two units plus half of the third one, when the unit is *tyve* ('twenty')" i.e., 20 + 20 + 20/2, precisely 50 (see Figures 1 and 2).

Figure 1: The 'three-times-twenty'.

Figure 2: The 'half-three-times-twenty'.

The process of counting described herein may present a counterintuitive challenge for students and individuals in general. When a certain algebraic knowledge is present, upon encountering algebraic expressions such as "half-three-times-twenty" individuals interpret the operator 'half' as applied to the entirety of the subsequent expression, namely, 'three-times-twenty', resulting in an expression akin to $(20 + 20 + 20)/2 = \frac{1}{2}(20 + 20 + 20)$. However, this interpretation actually diverges from the conventions of natural Danish language usage. Furthermore, as it will be evidenced in the excerpt from the mathematical discourse within an Italian sixth-grade classroom (refer to Section 5), while the concept of "half" may intuitively resonate and be swiftly introduced during primary education, it remains ensconced within one of the central challenges encountered by students and adults globally (from all over the world): the learning of fractions (e.g., Castro-Rodríguez et al. 2015). And this theme is intertwined with two issues, primary for this context: the presence of difficulties arising from the natural numbers bias, particularly concerning the density property of rational numbers; and the complexity posed by the multitude of interpretations and representations of fractions (Pitkethly and Hunting 1996). Acknowledging the pivotal role of the evolution of semiotic representations in advancing mathematical cognition (Duval 2006), Marmur et al. (2019) suggest how the use of signs and symbols denoting mathematical objects may precede the attribution of mathematical significance to them. In this case, this issue of the Danish use of $\frac{1}{2}$, which could be considered merely a "linguistic or cultural issue", actually structures thought. It will most likely not be the same for an Italian or a Danish child to hear 'half of three times twenty'. Yet both must know how to work out the following arithmetic expression: $\frac{1}{2}(20 + 20 + 20)$, and the thoughts of both can mutually enrich each other's learning process.

The QRS-system has now emerged. The relationships between concepts are relevant to learning and teaching mathematics. The cultural and semiotics aspects of mathematics cannot be ignored. And given that gazing at a single sign is no longer sufficient, the semiosphere emerges as a suitable lens for a mathematics education study.

For the sake of a clearer comprehension, a description must be given of the paradigm of mathematics education in which this research is embedded, i.e. how learning is considered.

2 Learning mathematics as a development of mathematical discourse

Traditional educational studies conceptualize *learning* as the "acquisition" of entities such as ideas or concepts, no matter if the term "acquisition" is interpreted as passive reception or as active construction. The acquisitionist approach relies on the idea of cognitive invariants that cross cultural and situational borders. Consequently, the theories that come from the acquisitionist tradition are geared toward finding and investigating what remains constant when the situation changes. And yet, as argued by many authors (e.g., Andrews 2010; Artigue 2008; Cole 1996), human learning is too dynamic and too sensitive to ongoing social interactions to be fully captured in terms of decontextualized mental schemes, built according to universal rules. The disillusionment with *acquisitionism*, although greatly precipitated by the advent of digital recording, began, in fact, prior to the advances in data-collecting techniques. Cross-cultural and cross-situational studies that had proliferated since the first decades of the twentieth century systematically undermined acquisitionist claims about developmental invariants. Their results drew researchers' attention to the social and cultural contexts of learning. Learning is then defined as "a social, cultural, and historical activity" (Erbilgin and Arikan 2021). The foundations of a new learning conceptualization, in a mathematics field, ascribed to this change of perspective are: mathematical knowledge is created and agreed to by a community because of a need to explain, interpret, communicate or explore (Hersh 1979); learning is continuous, evident in every aspect of our lives, there is no one final "knowledge" in any domain (Vollrath 1994); and participating in an activity, including a social activity or personal reflection, impacts on our knowledge, understanding, and interpretation of the world, hence results in learning (Engeström 1999; Vygotsky 1978). According to this new perspective, Sfard (2001: 25) defines learning as "a special kind of social interaction aimed at modification of other social interactions." Thus, rather than looking for those learner's properties that can be held responsible for his/her constancy in cognition, a framework that allows to stay tuned to the interactions from which the change, the transformation, arises is needed, without rejecting the acquisition metaphor, but rather subsuming this more traditional outlook, while modifying its hidden epistemological infrastructure. Psychologists, sociologists, anthropologists, and cultural studies scholars now conceptualize developmental transformations as changes not in individuals, but rather in what and how people are doing, and they claim that patterned collective activities are developmentally prior to those of the individual. Sfard (2007) calls this a *participationist* perspective.

One basic principle of the participationist perspective is the overcoming of the thinking-communicating dualism. As in Wittgenstein (1953: 108), participationists believe that “Thought is not an incorporeal process … which it would be possible to detach from speaking.” And whatever the form in which thought is expressed, image, word or other semiotic resource, “thinking is a special case of the activity of communicating” (Sfard 2001: 26).

> Consequently, thinking stops being a self-sustained process separate from and, in a sense, primary to any act of communication and becomes an act of communication in itself, although not necessarily interpersonal. To stress this fact, I [Sfard] propose to combine the terms *cognitive* and *communicational* into the new adjective *commognitive*. The etymology of this new word will always remind us that whatever is said with its help refers to phenomena traditionally included in the term *cognition*. (Sfard 2007: 570, italics in original)

And further,

> Our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response. If so, becoming a participant in mathematical discourse is tantamount to learning to *think* in a mathematical way. (Sfard 2001: 26, italics in original)

Therefore, in accordance with the participationist perspective, *learning mathematics* is defined as a development/evolution/change of mathematical discourse.

Sfard (2008) means by *discourse* the different types of communication that bring some people together while excluding some others. Mathematical discourse is an example of a particular type of discourse (thus thinking).

Mathematics discursive development is characterized by Sfard in identifying *transformations* in each of what she defines as the four discursive characteristics: the use of words characteristic of the discourse, the use of mediators, endorsed narratives, and routines (see Sfard 2008 for further details).

Two types of learning exist Within the *commognitive framework*: *object-level learning*, which expresses itself in the expansion of the existing discourse, attained through extending a vocabulary, constructing new routines, and producing new endorsed narratives – transformations that can be achieved by the students on their own, without the help of a more experienced participant; and *meta-level learning*, which involves changes in meta-rules of the discourse – for these transformations some special conditions are necessary, one of which is the fact that such learning can only take place collectively and with the support of the expert participant. Also Bartolini Bussi (1998), with her definition of *mathematical discussion*, argues in favor of a construction of mathematical discourse as theoretically understood as impossible without an experienced participant. Yet Sfard adds a major condition: the opportunity for meta-level learning arises when learners encounter a discourse that is *incommensurable* with their own. That is, on a semantic level, when in the same discourse, the same word is used in different ways. Sfard (2007) echoes this concept from Rorty (1979 after Kuhn 1962), who by the adjective "commensurable" means the ability "to be brought under a set of rules that [tell how to reach a rational agreement that] would settle the issue on every point where statements seem to conflict" (Rorty 1979: 316).

For the sake of a better understanding, the word "conflict" deserves to be explored.

Conflicts of different kinds, from epistemological to cognitive, mark possible evolutions in the way mathematical understanding develops in students, and sometimes they also appear, at another scale, in mathematics teachers' professional development. This is an old repeated story in mathematics: from the discovery of irrational numbers in the Greek scientific world to recent findings about deterministic chaos, bafflements are a usual way, according to which old paradigms are broken and mathematics and its knowledge(s) go on in their development. In this way new, mathematical knowledge is often generated through conceptual and cognitive discontinuities: they challenge mathematical sense-making for students and for those knowledgeable of the discipline, generally creating what many authors call *conflicts* (Sfard 2008; Tall 1977; Tall and Schwarzenberger 1978), what others call *contradiction* – which refers to the "accumulating structural tensions within and between activity systems" (Engeström 2001: 137) – and what still others call *obstacles* (Brousseau 1997), often accompanying the substantive with adjectives, like *cognitive*, *epistemological*, etc., according to their main focus, on students' or teachers' processes or on the discipline content. For example, in Brousseau the notion of epistemological obstacle is introduced as follows:

> Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. They can be found in the history of the concepts themselves. (Brousseau 1997: 87)

Hence, an obstacle is "a piece of knowledge or a conception, not a difficulty or a lack of knowledge" (Brousseau 1997: 99). As such, an obstacle can be revealed by learners' errors, but it must not be confused with errors, which are the effect of the obstacle, namely, "of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (Brousseau 1997: 82). However, an obstacle has also a cognitive nature, insofar it entails the necessity of a fresh way of thinking that the new knowledge would require but apparently is not coherent with the previous one and encounters difficulties to be activated. Difficulties in an obstacle may be particularly subtle since, at first glance, the relationship between the old and the new knowledge seems a contradiction between the two, but generally it is not so: the new frame simply enlarges the old one, putting forward a new standpoint, which allows one to embrace the previous one in a new setting, which does contradict the older one in case the older framework is still used. Sfard thus defines what she calls *commognitive conflict* as follows:

> a situation in which communication is hindered by the fact that different discursants are acting according to different meta-rules (and thus possibly using the same words in differing ways). Usually, the differences in meta-rules that are the source of the conflict find their explicit, most salient expression in the fact that different participants endorse contradicting narratives. (Sfard 2007: 374)

Because the same words are used in different discourses, incommensurability may be invisible to discourse users. Instead, they may perceive an apparent *incompatibility* of narratives. But these narratives are not talking about the same thing. For example, Euclidean geometry is incompatible with hyperbolic geometry, but it is just incommensurable with the geometry necessary to describe "strange" objects like the Sierpiński triangle (Apkarian et al. 2019).

Thus, the term *conflict* indicates when a piece of new knowledge meets the old, when the latter reveals to be inadequate to solve a fresh problem (epistemological side); the new knowledge is incommensurable but not contradictory with the old and consequently requires new ways of reasoning (cognitive side).

Sfard (2007: 574), however, specifies that "the notion of commognitive conflict should not be confused with the acquisitionist idea of cognitive conflict, central to the well-known, well-developed theory of conceptual change." She lists three reasons for this distinction:

– the first is within the locus of the conflict, that is, by contrasting the truth-falsity of a concept – of which the world is the arbiter – with the idea of incommensurability between discourses;
– the second is in their significance for learning, that is, from "an optional pedagogical move, particularly useful when students display 'misconceptions'" (2007: 575), to an indispensable source of meta-level mathematical learning;
– the third is in the way the conflict is to be resolved, that is, moving from the principles of incompatibility and noncontradiction – two supposed contradictory narratives are also mutually exclusive, with a common criterion to reject or endorse and label one as true – to a conflict resolution as making sense of other people's thinking-talking about the world with "a gradual acceptance, 'customization', and rationalization – figuring out the inner logic – of other people's discourses" (2007: 576).

3 The cultural conflict

Consider again the example of the Danish numeral system.

In mathematics, an algebraic structure is a non-empty set together with a family of operations (such as addition and multiplication), and (sometimes) relations. With respect to the tens with an odd first digit in the Danish numeral system, it is evident a *different* interpretation of the *usual* operations of addition and multiplication. But these are just conventions. All mathematicians know that there are many algebraic structures, all equally valid. The natural numbers ($\mathbb{N}$; +, ·) and rational numbers ($\mathbb{Q}$; +, ·) have *canonical* algebraic structures with a sum and a product. In reading these symbols alone, no one would doubt how addition and multiplication behave, thinking immediately to their natural algebraic structure, where multiplication is distributive over addition, addition and multiplication are associative and commutative, etc.; in short, where all the *normal* rules apply. However, the very fact that we use the terms "natural," "canonical" or "normal" is a cultural aspect.

The cultural difference lies in the fact that if someone speaks of the addition and multiplication of rational numbers, without specifying anything else, it is our instinct (for Italian people, for example) to assume that we are working in the canonical algebraic structure. This is simply because we are in a culture that privileges this structure, and our calling it the "natural"/"usual" structure of $\mathbb{Q}$ is perhaps a symptom that it comes most naturally to us. But possibly, finding this wording for multiples of ten in Danish, it occurs that probably if mathematics had been born in Denmark, then it is not this structure that we would call "canonical" or "natural," but rather a structure in which a rational number ($\frac{1}{2}$) acts on a sum of natural numbers $(20+20+20)$ in a different way (for example, $20 + 20 + 20/2 = 2 \cdot 20 + \frac{1}{2} \cdot 20 = 50$). This is certainly a valid choice of addition and multiplication, which does create another perfectly suitable algebraic structure, but it is evident (for us) that it belongs to a different cultural context. On hearing "half-three-times-twenty" in the absence of an explanation of the interpretation to be used, we would not understand. We would use the "usual" distributive property by obtaining $\frac{1}{2}\,(3 \cdot 20) = \frac{1}{2} \cdot\,(20 + 20 + 20) = 30$.

This is not a cognitive conflict, there is no misconception or misinterpretation. Instead, it is a difference in the use of operations, of algebraic structures. Nor is it just a matter of a commognitive conflict, in which incommensurable discourses foster a necessary acceptance and rationalization of the discursive practice of an expert interlocutor. Rather, it is something different. This is what I have defined as *cultural conflict*. The two previous definitions of conflict, which do exist and certainly contribute to learning, are no longer sufficient, but there is something further. This is where Lotman's semiosphere supports mathematics education researchers, serving as a theoretical lens that enables the significance of cultural aspects in mathematics learning-teaching, and providing a way of reading and interpreting them. Different mathematical texts acquire meaning and value through the asymmetrical structure of the semiosphere, in encounter and dialogue, in conflict and mutual translation.

The concept of cultural conflict does not alter the assumed research paradigm (see Section 2), but embeds it within the theoretical framework of the semiosphere. Table 1 defines the cultural conflict and describes the differences between the concepts of cognitive, commognitive, and cultural conflict.

Table 1: Comparison of concepts (adapted from Sfard 2007: 756).

Concept	Cognitive conflict	Commognitive conflict	Cultural conflict
Ontology: The conflict is between	the interlocutor and the world	incommensurable discourses	asymmetrical contexts
Role in learning	is an optional way for removing misconceptions	is practically indispensable for metalevel learning	is a transformative process through a redefinition of the problem and a critical self-reflection of the assumptions
How is it resolved?	by student's rational effort	by student's acceptance and rationalization (individualization) of the discursive ways of an expert interlocutor	by making explicit learner's *unthought* [but potentially activated by the encounter with foreignness]

4 Cultural semiotics and mathematics education

The growing interest aroused by semiotics in the field of mathematics education is purely justified not only because the "mathematics relies on an intensive use of different kinds of signs" (Radford 2001: 1), but also for manifold reasons – the following is a summary of three of the main reasons, all of them intimately interconnected, for a semiotic theorization of mathematics education:
- the role that signs play in cognition;
- the role of semiotics in the interpretation and construction of meanings;
- the existence of sign systems and the fact that they compose a certain unit in which they function and support each other, i.e., a culture (Uspenskij et al. 1998, in Rebane 2013).

Examples of theoretical approaches to mathematics education relying on semiotics are Arzarello's (2006) *Semiotic Bundle* and Bartolini Bussi and Mariotti's (2008) *Theory of Semiotic Mediation*, but several more exist. The relevance of semiotics in mathematics education is certainly not new (Presmeg et al. 2016, 2018; Radford 1998, 2001), nor is the use of the semiosphere as a theoretical lens for this field of research. In fact, it is from the very need to have a systematic way of linking theories in this field of research and to reflect on the networking process and its outcomes that the *Networking of Theories* paradigm was defined (Bikner-Ahsbahs and Prediger 2014). And here Radford (2008) suggests that a space for the networking of theories needs to be assumed, a meta-language for talking about theories and networking practices to build connections among them. Referring to Lotman (1990), Radford calls such a space a semiosphere, that is "an uneven multi-cultural space of meaning-making processes and understandings generated by [theories] as they come to know and interact with each other" (Radford 2008: 318). It is thus a way to renew theories of mathematics education field in different ways. The main function of the semiosphere here is to provide possibilities for each theory to engage in dialogue, thus creating different ways of fruitful connections, "such as deepening the identity of a theory, integrating different theories into a new one or just locally, or creating new kinds of research questions" (Bikner-Ahsbahs et al. 2010: 146).

In my PhD dissertation (Manolino 2021), however, I theorize that the semiosphere is not only relevant for leading to a "dialogue of theories" in our field of research, but rather that it is a critical theoretical lens to enable the study of learning processes and teaching of mathematics: the development of mathematical discourses. It provides a theoretical tool to approach cultural aspects of mathematical discourses, a much sought-after and hitherto missing feature.

Lotman's notion of *context*, in fact, challenges us as researchers to consider no longer just the individual sign, but its interaction with otherness. The context is not the outwardness of mathematical discourse, that which lies outside of it without having its specific properties, such as – it could be conceived – institutional contexts, school curriculum, beliefs, etc.

Indeed, there are already theories that study conditions and constraints, that is institutional and ecological dimensions, of mathematical knowledge, e.g., the *Anthropological Theory of Didactic* (ATD) of Yves Chevallard (1999) and the *Onto-Semiotic Approach* (OSA) of Godino et al. (2019). And a key tool for the analysis of the ecology, developed within the ATD, is the *scale of level of didactic codeterminacy* (see Figure 3). The specific levels relating to the structure of a given discipline are the lower levels of the scale. The higher, the *generic* levels, are common to the teaching of any discipline. The flow between levels reveals Chevallard's attention to cultural aspects.

However, according to Florensa and collegues (2018), there is "a strict separation" between the upper and the lower levels. The teaching problems are conceived as independent of the taught (or to be taught) content. Lotman's notion of *context* and the spatial dynamics of the semiosphere (Figure 4) can provide the methodology

Upper levels	*Lower levels*
Humanity	↓↑
↓↑	Discipline
Civilisation	↓↑
↓↑	Domain
Society	↓↑
↓↑	Sector
School	↓↑
↓↑	Theme
Pedagogy	↓↑
↓↑	Question

Figure 3: Scale of levels of didactic co-determinacy (Florensa et al. 2018: 5).

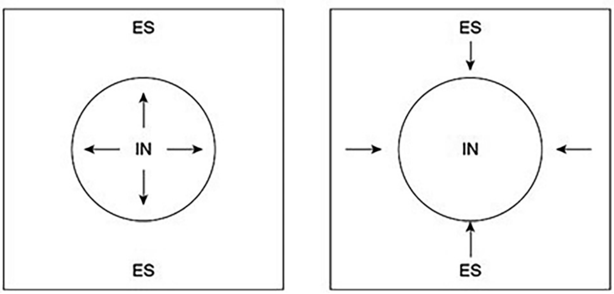

Figure 4: The spatial dynamics of the semiosphere (Lotman and Uspenskij 2003 [1969]).

to be used to analyze the conditions and constraints coming from the higher level of the scale and thus connect the two sets of levels.

Consider one more time the example of the Danish number system. It could be used in primary mathematics teacher education. In fact, prospective teachers are expected to be aware and know operations, properties, and meanings of natural and rational numbers in relation to our decimal positional system and early algebra and generalization processes. The evolution of the mathematical discourse of future teachers could then be channelled from a consideration of the spatial dynamics of the relative semiosphere. An inward movement is realized. The algebraic structure of the Danish system allows one to realize the importance of properties, the notion of unity, and how to introduce them to children. These are notions that are often taken for granted and there is a usual, ritualized, noncritical way of teaching them in school.

The algebraic structure of the Danish system is not only meaningful in itself, but when looked at in relation to the cultural context in which it is embedded (vigesimal system, different associative and distributive properties) it can help, for example, a prospective teacher to grasp the significance of the numeral system's properties and operations. Not so much because perhaps there will be a child in the classroom who will be using that algebra and the teacher will need to be able to teach it, but to avoid taking its "naturalness" for granted.

5 A mathematical discourse on the "half"

To delve into the application of the theoretical framework, I present an excerpt from a mathematical discourse. This took place at the end of May 2023 (the school year typically concludes in early June) in a sixth-grade classroom in Italy. To uphold individuals' privacy concerns, we denote here the students as S, numbering them based on the sequence of their contributions to the discourse. Four teachers and one researcher were present in the classroom. Only three teachers participate in this excerpt, denoted as T followed by a corresponding numeral, while the researcher is designated as R. This excerpt is part of a lesson study cycle conducted at the school (Arzarello et al. 2023; Manolino 2024).

A group of students, following a task on interpreting a graph, was tasked with discussing their resolution procedures. S1 stands before the class, displaying on the board what they had written on the protocol. The notation "1.6 years" in the table crafted by the group immediately catches attention (see Figure 5).

The discussion ensues, starting from their interpretation of "1.6 years" as "a year and a half" (line 23, in Table 2) based on the premise that "A year has 12 months" (line 7) and "6 is half of 12" (line 73). The evident challenge in interpreting the semiotic representations of "half" becomes apparent. Various forms of representation are mentioned, including written expressions, verbal descriptions with different formulations, over-and-under fraction, decimal numbers, whole numbers, and also cake divisions.

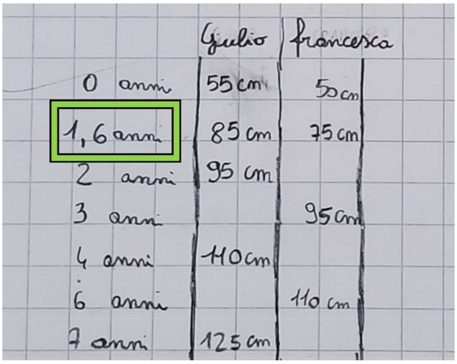

Figure 5: Protocol excerpt displaying students' interpretations. The notation "1.6 years" prompts the discussion.

Table 2: A mathematical discourse in a sixth-grade classroom in Italy.

Line	Mathematical discourse
4	**T1**: Why did you write 1.6?
7	**S1**: [Shrugs] A year has 12 months…
8	**R**: Okay! A year has 12 months
9	**Class:** Well, no… Actually…
	[**S1** shrugs with raised shoulders]
19	**T2**: And that 6 there, what does it signify? [The word used by the teacher T1 here is actually *vale*, which in Italian conveys a message of meaning but also of value.]
20	**S4**: Some months…
21	**Class**: Half. Half
22	**S5**: Half. [Here S5 use the word *mezzo* instead of *metà*, used before
23	**S2**: So, a year and a half
24	**T3**: How did you write "a year and a half." R2? Ah no, you wrote: "1 and a half," you wrote
25	**R**: Many wrote 1 and "a half" in words, or 1 and the fraction [gesturing, writing the fraction $\frac{1}{2}$ in the air]…
31	**S8**: I think writing 1.6 is correct
32	**R**: You think writing 1.6 is correct
36	**S7**: But it's not a number. [He means a "whole" number]
37	**T3**: As a decimal number, how would you write "a year and a half"?
38	**S3**: As a decimal number…
39	**S8**: At 9 turns the ten, so you go…
40	**T3**: Think about how you write the tens
41	**S8**: … move forward. And then comes the 7. It's as if the ten is at most 12
46	**T1**: If you said "one and a half meters," how would you write it?
47	**Class**: 1.5
48	**T1**: Instead, "one and a half years" 1.6
49	**Some**: Yes! Yes
50	**Some**: No! No. No. No
52	**T3**: And an hour and a half?
53	**S10**: Well, it's always the same
54	**S12**: An hour and a half… and 30!
55	**S11**: There's a half-hour
66	**S2**: There's one and a half … It's just that you have to write … I mean one year and a half
67	**R**: "And a half"? Where does it write "and a half"? It says 1.6
69	**S2**: Well, but it reads "one year and a half"
72	**S2**: Well, but the 6 is wrong
73	**R**: … Why is it wrong? Because they say so. Because 6 is half of 12. There are 12 months in a year
77	**R**: She says, "In any whole, the half is always point 5"
81	**S7**: I mean, if you take a cake and divide it into 12 slices, and you have to consider 6, you don't take 5, because then you'll have 7 left over. I mean, you always have to take half
86	**S10**: Because… I mean, if we take one, which counts as a year, the 5… I mean, we should put 5, not 6, because we have to divide one in half, that is… I mean… Too many I means
87	**R**: So, that's what she said

Table 2: (continued)

Line	Mathematical discourse
88	**T2**: One is half of 1… [She leaves the phrase open: "the other is…"]
89	**T3**: Half of 1 is always 05 as she said
95	**R**: He says no, because it depends on how many parts you divide the whole into
96	**S7**: I mean, I don't know if it's a no, eh. I don't know if it's yes or no. Just in my opinion, it seems right…
97	**S13**: In my opinion, you can choose!
106	**S13**: In my opinion, if someone sees the graph without it being explained, they won't interpret it well. I mean … But, in my opinion, what matters is that whoever made the graph can understand and explain it, even if they wrote something different
108	**Class**: You can choose. You can choose
114	**S10**: I mean if there's a conference…
116	**S10**: It doesn't matter if you understand it. I mean, if someone's there in front of you … You have to explain it
118	**S9**: But **S10**, in a conference, do you just open an image and let them look at it?
119	**S10**: No, but the doubt arises like it did for us if it's wrong
120	**S9**: Okay, but then you explain

From this discourse, it is evident that, although it was not the central theme of the task, educators found it essential to dedicate over 9 min of the 55 min allocated for the lesson to address this student misconception. This mathematical discussion was not initially planned within the lesson study. The semiotic representations that teachers sought to evolve in the students' discourse revolve around the concept of "half." The conflict remains primarily commognitive; however, interpreting the discourse necessitates immersion within its cultural context: students adhere to classical representations of rational numbers (such as dividing a cake – lines 81–86; the various Italian terms denoting the concept – e.g., lines 21–22) which, upon encountering the representation of the concept "as a decimal number" (lines 37–38), precipitates conflict. Here, the dense structure of rational numbers contrasts with the discrete structure of natural numbers. Moreover, students struggle to represent a concept they have in mind. They know what they want to convey but struggle to represent it. Indeed, many, across different working groups, "wrote 1 and 'a half' in words" (line 25). If there had been an opportunity to encounter a different cognitive structure of the concept of half, the discourse could have broadened. Employing representations that are familiar conditions individuals to revert to a pre-existing interpretation of the concept. This is particularly true for basic concepts like "half," known since childhood. Encountering something unanticipated that disrupts one's existing structure could instead foster a new interpretation. Shifting the conflict to a cultural plane would have been desirable to face it.

6 Conclusions

It is not a question of nullifying the cultural conflict, but exploiting the concept of asymmetry to make sense of mathematical discourse. By encouraging people (students and teachers, but also researchers and scholars) to move into a different cultural paradigm, people can gain awareness of what Jullien (2005) defines as *unthoughts*, i.e., those aspects that escape people's consciousness when they are immersed in their own culture. It is not so much a matter of understanding foreign cultures, thought of as homogeneous spheres with marked boundaries, but of an "interaction with foreignness" (Welsch 1999, in Barton 2008). Meeting foreign cultures leads to looking at one's own practices.

With this paper, the author aims to prompt reflection within the field of mathematics education, particularly regarding the theoretical research context. Understanding how to move beyond a local interpretation of signs and comprehend them within their cultural context appears to be increasingly essential, even within a domain like mathematics, traditionally viewed as universal, aseptic, and uniquely interpretable. Much work is needed, but commencing from educational action and the observational capacity of both researchers and educators, illuminated by a well-defined theoretical framework, undoubtedly serves as a foundational starting point.

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