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Semiosphere lens for mathematics teacher education

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Abstract
This paper discusses the lack of compulsory practices for mathematics teachers' professional development in Italy, as well as the challenges that arise when trying to implement changes in teaching and learning practices. The 2018 OECD Teaching and Learning International Survey shows that merely providing a body of knowledge may leave some teachers uncertain about the necessary steps to enact meaningful changes. Barriers such as scheduling conflicts and insufficient incentives further hinder participation in professional development programs, while transforming teaching practices remains challenging due to cultural, psychological, pedagogical, and disciplinary issues. Scholars provide the epistemological legitimacy of the taught knowledge, but there is a need for sustainable mathematics teacher education, as well as research into cultural and social aspects to enable mathematics teachers' critical reflection. This study uses Lotman's Semiosphere lens to analyse data collected during Lesson Study experiences in mathematics, which is a Chinese teacher professional development methodology that can trigger critical reflection. Examining two case studies involving grade 5 and grade 12 students in the domains of probability and combinatorics, the research illustrates the essential activation of teachers' reflective practices concerning both the mathematical concept itself and instructional methodologies within this culturally sensitive practice for professional development. The presented case studies serve as examples illustrating the theoretical positioning. As a research finding, mathematics teacher education that also focuses on cultural aspects has a more sustainable and fruitful outcome.

Keywords Semiosphere · Teachers education · Lesson study · Combinatorics · Probability

1 Introduction

In Italy, there are no compulsory contents or practices for mathematics teachers' professional development. The results of the 2018 OECD Teaching and Learning International Survey,

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which aims to examine the level of professionalism in teaching, show that teachers' education has a positive impact, a change, on teaching practice. However, "prescription does not motivate all teachers to change their practices, […] providing only a body of knowledge might leave some teachers not knowing what direction they should take to enact changes" ([18], p. 2). Furthermore,

1. only 44% of teachers affirm to participate in training based on networking, even if they also report that professional development based on collaboration and collaborative approaches to teaching is among the most impactful for them;
2. around half of teachers and principals report that their participation in the professional development available to them is restricted by scheduling conflicts and lack of incentives;
3. changes in teaching and learning practices "are a challenge", due to the fact that we are dealing with cultural, psychological, pedagogical ([33], p. 230) and disciplinary issues.

Universities and researchers are responsible for addressing such issues and proposing appropriate measures. Scholars provide the epistemological legitimacy of the taught knowledge. Indeed, any human activity "has its own scholars, held 'to know best' than the rest of the people in the little world where they belong" ([22], p. 76). This is particularly pronounced in the Italian academic tradition where general pedagogy aspects are traditionally separated from problems of teaching science and mathematics. However, knowing mathematics is different from knowing how to teach mathematics. There is a need for sustainable mathematics teacher education and research into cultural and social aspects to enable mathematics teachers' critical reflection. It is a scholars' accountability to comprehend how to bridge the epistemological, cultural, social, and cognitive dimensions of mathematics and its pedagogy.

I address the issue using Lotman's [47] Semiosphere lens.

In my research hypothesis, there is a call for a theoretical re-evaluation when addressing the challenges associated with the teaching of mathematics. The phenomena delineated in this paragraph are based on the fact that teaching–learning of mathematics constitutes a "cultural activity" [17, 33] but also underscore its integration within a broad complex system.

As articulated within the framework of the Anthropological Theory of Didactics [22], moving from mathematical knowledge to its teaching–learning processes is by no means straightforward. The concept of *didactic transposition* intricately involves the process of deconstructing and reconstructing various knowledge components, with the overarching aim of rendering it "teachable". This transformative endeavour is orchestrated by a diverse array of agents, collectively constituting the *noosphere*, i.e. the sphere, the space in which the transposition is thought. These agents encompass politicians, mathematicians ('erudites'), and members of the educational system, including teachers. The didactic transposition unfolds within historical and institutional contexts that are not always straightforward to discern. Contemporaneously, a *cultural transposition* [53] is necessary, constituting a process of deconstruction that realizes a system wherein "different cultural backgrounds generate possibilities of meanings and of mathematics education perspectives[, organising] the contexts and school mathematics practices in different ways" ([51], p. 571). This underscores the caution that mere translation (the adoption of practices from different contexts) is insufficient. Such an approach fails to reconstruct the cultural system in which practices derive meaning and does not rebuild the semantic field of the terms used. The emphasis, therefore, lies not in merely importing diverse practices but in understanding varied educational practices to delve deeper into the meanings of one's own practices [28]. Consequently, a theory is requisite that not only considers institutional and social aspects but also incorporates signs and meanings. Lotman's Semiosphere is particularly fitting in this regard, serving as a theory of a global

semiotic space that, in its unity, imparts significance to every signifying act. It consistently aligns and expands the conceptualisation of teaching mathematics as a "laboratory" for constructing meanings and senses [7], within the semiotic perspective of Mathematics Education [58]. By connecting and interweaving meanings, even those culturally diverse, into a unified network, it links them together. Therefore, by merging the existing concepts of didactic transposition, cultural transposition, and utilising the theoretical lens of the Semiosphere, emerges the possibility of implementing an educational and epistemological transposition of mathematical practices. This incorporates, in an educational form, profound epistemological changes/differences found in the history of mathematics or in different cultures. An example, addressed in the following in this paper (see LS1), is the mutual construction of meaning between the concept of fractions and the classical definition of probability when introduced in Italian primary schools, where teachers may not be entirely confident with these theoretical notions, yet they are tasked with deciding solely on how to implement this instruction. The purpose of this paper is to demonstrate the efficacy of integrating didactic transposition into a collaborative teacher education framework, such as Lesson Study. Within this context, the semiotic dimension becomes manifest through the deliberate articulation of meanings via teacher discourse. Employing the theoretical framework of the Semiosphere facilitates a comprehensive examination of all the culturally stratified layers of meaning of the considered mathematical knowledge and instruction.

Radford [60] had previously employed the Semiosphere in the field of Mathematics Education in his comments on networking approaches to theories. However, my application involves leveraging the Semiosphere from an alternative standpoint: to analyse the transformation of teachers' discourses. A breakthrough from my study is precisely in applying Lotman's conceptual framework to the professional development of teachers.

My study is based on a semiotic-cultural analysis of data, collected during Lesson Study experiences in mathematics [13], i.e. a Chinese teacher professional development methodology that can trigger a critical reflection. If observed through the theoretical lens of Cultural Transposition [53], Lesson Study as a foreign methodology questions the teaching intentionalities of Italian mathematics teachers. The analysis of two experiences, carried out respectively with grade 5 and grade 12 students in the areas of probability and combinatorics, are paradigmatic for this study and demonstrate the essential activation of reflective practices around the mathematical concept itself as well as teaching practice.

2 Rationale and literature review

Recent literature on teacher professional development highlights collaboration among teachers [3, 9] and semiotic analysis of students' thinking as key factors for supporting teacher learning (e.g., [32, 50, 78]). Studies by Gonzalez and Maxwell [34] and Franke et al. [31] have shown that teachers who have knowledge of students' diverse learning approaches, problem-solving strategies, and can adapt their instruction accordingly, can provide inclusive, innovative and appropriate teaching activities in the classroom [57]. However, in many European countries (and beyond), teacher education is often disregarded or provided in a manner that does not fully address teachers' actual needs (see the 2018 OECD Survey). While good progress has been made in improving teaching–learning processes, their widespread use and triggering of a capacity for their critical and intentional use remains a challenge [66], not to mention studying to improve culturally sensitive understanding of educational processes [70]. A number of suggestions have been proposed for enhancing the professional

development of teachers beyond the frontal lecture, the few-hours workshop, or the ubiquitous sharing-chatting on good classroom practices [63]. In the United States, since *The Teaching Gap* [71, 72] showcased the videotaped lesson of the Third International Mathematics and Science Study (TIMSS) (from the United States, Germany and Japan), Japanese Lesson Study has been highlighted as a milestone in contributing to a supportive way to improve in-service teachers' classroom instruction [26, 29].

Thus, there is a need for theoretically based research about the way cultural aspects influence students' learning in mathematics [12] as well as mathematics teachers' critical reflection during their professional development experiences [70]. Teachers' professional development studies and courses that take cultural factors into account are sparse (e.g., [52, 54]), although their importance is recognized [35, 40, 73]. To address this issue in Mathematics Education, several researchers have suggested a cultural approach that can lead to more effective educational choices in teacher professional development and significant changes in their teaching practice (e.g., [52, 54, 74]). This approach aligns with the cultural approach to mathematics education developed by Bishop [17] and D'Ambrosio [23]. Bartolini Bussi et al. [11] found that, encountering educational practices adopted in other cultural contexts, teachers can improve their own teaching practices by reflecting on their educational intentionality. Mellone et al. [53] developed the construct of *Cultural Transposition* (CT) as a way to decentralise teaching practices in a specific cultural context by introducing teaching practices from different cultural contexts. A fundamentally different process from the mere importation of foreign educational activities or comparative practices. This can lead to reflection and development of awareness for educators and teachers [8]. While the development of teachers' awareness is well-recognized in research on Mathematics Education as a need to improve teaching professionalism, it is not as well-implemented in school contexts. In recent years, there have been various CT experiences in different school contexts, and some professional development paths have been designed and implemented to reduce the gap between research and practice. Examples of such experiences and paths include those developed by Bartolini Bussi et al. [11] and Mellone et al. [52–54], as well as studies by Di Paola and Buttitta [25].

Lesson Study (LS) is thus one of the main sources for researchers, that holds together a cultural perspective and collaboration between teachers, to gain insight into teachers' attitudes towards working and learning, their mathematical knowledge for teaching, and teaching practices [38]. LS is a collaborative model for teachers' professional development [13]. LS is usually modelled as a cycle and it is structured in the following phases: (1) Goal setting: formulate a "research" theme of common interest to teachers around a topic to be taught, (2) Planning: collaboratively prepare a Lesson Plan to implement the research lesson; (3) Implementing and Observing: one group member teaches the lesson while other members presentially observe; (4) Discussing: members discuss the data collected during the observation and revise the lesson. Then the cycle could be repeated multiple times. This structure has many variants with different features according to diverse cultural aspects (i.e., [67]).

LS originated in Japan around 1870 to answer professional development needs for qualified teachers [39], and it has been in place for more than a century in both China and Japan nationwide [21, 30]. The Lesson Study was introduced in China to foster, through peer-to-peer interaction, the in-service professional development of hundreds of thousands of teachers needed for Mao Zedong's literacy project in the 1950s (see the history of the *Great Proletarian Cultural Revolution* (无产阶级文化大革命)). Only recently LS has been gaining momentum on a national scale (and no longer through the intervention of individual researchers) in Western nations as well.

Fig. 1 The "constellation" work: distribution of the compulsory 30 h of French teachers' professional development

An interesting example is the French education system, which just in 2020 approved, for now for primary schools only and within French language and mathematics subjects, a 6-year ministerial mandate for an innovative professional development model based on group work and peer instruction:

> Professional development is grounded in the local teaching context (classrooms become the sites of professional development), and it articulates the exploitation of theoretical resources and variations in the classroom, allowing teachers to rely on research findings to find appropriate responses to their needs and those of their students (from French National Professional Development Plan[1]).

The plan radically changes the French in-service teachers' professional development. The official guide to the plan announces: "a new approach to in-service professional development, incorporating an accompanied reflective analysis: working within a small group [the *constellation*] of 6–8 teachers, led by a local trainer and set up as close to the classroom as possible" (see footnote 1). The constellation is declaredly inspired by the model of *lesson studies*. The guide says: "The sequence developed collectively with external help is then implemented in a classroom by one or more teachers. The other teachers in the group observe this implementation, in particular, to assess the effectiveness of student learning. The next step is to discuss the observations to identify what has been learned and what can be improved". The schedule for each constellation is 30 h per year, distributed as illustrated in Fig. 1. Nevertheless, for the moment, the ministerial proposal is not fully accepted by French teachers. The web page of a French school labour union reports: "The lesson study model has been proposed a lot by the OECD since Pisa 2015. In Japan, volunteer teachers form groups that jointly design a lesson. One of the group members implements it in front of his colleagues. After observing and analysing the session, the teachers meet again: the one who led the session speaks first, then his colleagues each present their analysis in turn. This model is proving to be effective, but in a society and educational system totally different from ours." (see footnote 1)

[1] https://www.sgen-cfdt.fr/actu/communique-formations-constellations/.http://50.sgenbn.fr/formations-en-constellation-plan-francais-maths/.http://www.cafepedagogique.net/LEXPRESSO/Pages/2020/06/29062020Article637290123216351278.aspx.

Since 1999, indeed, a great deal of research focused on the dissemination of Lesson Study around the world (i.e., [20, 38, 44, 59]). Thanks to Catherine Lewis' research [42] and Makoto Yoshida's doctoral thesis [79] the Japanese 授業研究[2] (*jugyoukenkyuu*), known in the Western world as the Japanese Lesson Study, has been introduced in the United States, creating an Anglo-American tradition. Only then it has been adopted and adapted globally, gaining the educators' and researchers' interest worldwide to promote collaboration and cooperation between teachers [1, 6, 11, 37, 77].

> With the high reputation in practical features such as job-embedded and teacher-oriented but student-learning-focused [43], lesson study has been well recognized as one of the most effective [professional development] ([36], p. 202).

A glimpse of the global impact can be gained by peeking at the World Association of Lesson Studies (WALS) activity, which organizes yearly conferences with teachers and researchers *deeply committed to refining our collective understanding and execution of lesson study* (quoted from the web page), and at the journal IJLLS, that is dedicated to research on Lesson Study.

In the Italian experience, an essential emphasis is given to the cultural context in which the teaching–learning is situated. This is because LS was introduced as a teacher education proposal that, by leveraging cultural differences, stimulates teachers to reflect on themselves and their own habits (e.g., [8]). LS is usually introduced by researchers to teachers, and emphasis is on the important role of the researcher as a cultural mediator [56]. In Italy, as a CT practice, the researcher does not mediate cultural adaptation a priori but uses it educationally as a stimulus for teachers' development [61]. Recently several studies have focused on analysing teachers' discourses as a window on teacher learning (e.g., [24]). This is in line with Sfard's [69] view of *learning*. From a participationist perspective, Sfard [69] means by *discourse* the different types of communication that bring some people together while excluding others. An example of a particular type of discourse (thus thinking) is the mathematical discourse. Learning mathematics is so defined as a "development" of/a change in mathematical discourse. Mathematics learning development is characterised by Sfard as the *transformation* of discourses, which can be identified through a semiotic analysis (see [69], for further details).

3 Theoretical framework

Teachers' discourses within specific collective practices of professional development (i.e., Lesson Study) are the unit of analysis of my research. It is indeed within collective practices that people unconsciously reproduce and alter different cultural forms of representation and signification [65]. In particular, Lesson Study, embedded in a Cultural Transposition perspective, is here analysed as the place where otherness of a different cultural paradigm is experienced to gain awareness of what Jullien [41] defines as *unthoughts*, i.e. those aspects that escape people's consciousness when they are immersed in their own culture. Meeting foreign cultures leads to looking at one's own practices. It is not so much a matter of understanding foreign cultures, thought of as homogeneous spheres with marked boundaries [5],

[2] *Nomen omen*. Concretely verifiable through any web translator, *kenkyuu jugyou* means *lesson study*, "and refers to the lessons that teachers jointly plan, observe and discuss. *Jugyou kenkyuu*—using the same two words in the reverse order—means *lesson research*, and refers to the process of instructional improvement of which the research lesson is the core piece" ([42], p. 4).

but of an "interaction with foreignness" (Welsch, 1999, in [14]). Every culture indeed needs another culture to define its own essence and limits.

3.1 Semiotic of culture: the semiosphere

A theoretical lens to read these contents exists in the field of the Semiotics of Culture and it is offered by Lotman [47]. I adapted this lens to the field of teacher education in mathematics with a systemic perspective, following the work of Radford [60] started in the broad realm of networking of theories.

Lotman argues that traditional scientific approaches have led to considering *the whole* as the sum of its parts in an ontological way, while division into parts is only a heuristic necessity. In fact, no part taken separately can really function: "they function only when they are immersed in a semiotic continuum filled with semiotic structures of different types and with different levels of organization" ([46], pp. 42–43). He calls this continuum a *Semiosphere*, by analogy with the concept of a biosphere introduced by Vernadskij. The semiosphere is the space outside of which "even the mere existence of semiosis is impossible" ([46], p. 44).

To read and then manage the relationships, the webs of significance that exist within the whole mathematics teachers' Semiosphere, their discourses are considered *texts*, the textualised culture, that is the focus of analysis of the Semiotics of Culture theory.

Two main features characterise the Semiosphere: the need for a *topological organization* and delimitation, and the characteristic of *structural unevenness*.

The Semiosphere is indeed always surrounded by an extra-systematic space or one belonging to another semiotic sphere, therefore it must manifest a form of homogeneity and individuality or semiotic personality, a characteristic trait of a whole, of a collective term. Therefore, in order to describe itself, the semiosphere resorts to dynamic spatial modelling, i.e. subject to continuous modification, and in any case dependent on the position of the observer. The dimensions that distinguish the models, schematised as in Fig. 2, are:

- *comprehensiveness*, i.e. the types of partition of universal space (does the semiosphere occupy all of the universal space or is it only a part of it?);
- the *dimension of universal space* (does universal space have known and, therefore, controllable boundaries or does it evolve and potentially expand infinitely?);
- *orientation* (within the overall space, in which direction is the semiosphere oriented?).

The second feature of the Semiosphere is mandatory internal irregularity and *unevenness* (неравномерность, [2]). In Lotman's words is written: "the structure of the semiosphere is *asymmetrical*" ([47], p. 127).

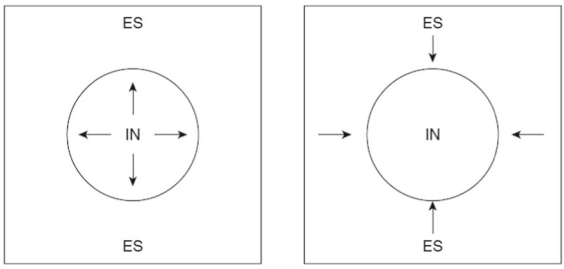

Fig. 2 The spatial dynamics of semiosphere ([45], p. 105)

> Asymmetry finds expression in the currents of internal translations with which the whole density of the semiosphere is permeated. Translation is a primary mechanism of consciousness. To express something in another language is a way of understanding it. And since in the majority of cases the different languages of the semiosphere are semiotically asymmetrical, i.e. they do not have mutual semantic correspondences, then the whole semiosphere can be regarded as a generator of information. Asymmetry is apparent in the relationship between the centre of the semiosphere and its periphery. At the centre of the semiosphere are formed the most developed and structurally organized languages, and in first place the natural language of that culture. [...] the semiosphere, besides the structurally organized language, is crowded with partial languages, languages which can serve only certain cultural functions, as well as language-like, half-formed systems which can be bearers of semiosis if they are included in the semiotic context. Compare the latter with a stone or a strangely twisted tree-stump which can function as work of art if it is treated as one. An object will take on the function ascribed to it ([47], p. 128).

It increases the complexity of understanding the topological organisation of the Semiosphere:

> A semiotic space is characterized by the presence of nuclear structures (more often several) with an explicit organization and a more amorphous semiotic world, which gravitates toward the periphery, and in which the nuclear structures are immersed ([46], p. 49).

There occur a number of what Lotman [47] refers to as 'invasions', variously affecting the internal structure of the semiosphere. In any synchronic section of the semiosphere—such as the taking of a photograph—there are different languages in conflict at different stages of development, some texts immersed in not their own languages, with the codes to decipher them entirely absent. To detail this heterogeneity of the semiosphere, Lotman uses the metaphor of a museum hall:

> imagine a museum hall where exhibits from different periods are on display, along with inscriptions in known and unknown languages, and instructions for decoding them; besides there are the explanations composed by the museum staff, plans for tours and rules for the behaviour of the visitors. Imagine also in this hall tour-leaders and the visitors and imagine all this as a single mechanism (which in a certain sense it is). This is an image of the semiosphere ([47], pp. 126–127).

Then, however, it should be remembered that all elements of the semiosphere are in dynamic, not static, correlations whose terms are constantly changing. In short, elements that refer to subjectivities that are also very distant from each other and scarcely translatable, but which are placed in a reciprocal relationship of dynamic tension, come to generate new traits and ways of meaning. A double tendency thus characterises the semiosphere: that of the elaboration of metastructural self-descriptions, and that of continuous reformulation and "reshuffling". While the former increases the rigidity of internal structures and slows down the development of the semiosphere, the latter can lead towards what Lotman [48] calls *explosions*, i.e. creative phenomena, dynamic tensions between highly conflicting moments.

Although Lotman's description of Semiosphere's structures is not ended at this stage, it is no longer useful for the present work of analysis to introduce other details. Rather, my research demands a networking approach [16, 60] and the contextualisation of the Semiotics of Culture within the field of Mathematics Education research. As outlined in the Introduction,

this networking of theories involves the Semiosphere and the frameworks of cultural and didactic transpositions. This approach fosters a semiotic perspective directed towards a shared concept: conflicts.

3.2 Discourses and conflicts

Conflicts of different kinds, from epistemological to cognitive, mark possible evolutions in the way mathematical understanding develops in students, and sometimes they also appear, at another scale, in mathematics teachers' professional development. This is an old, repeated story in mathematics: from the discovery of irrational numbers in the Greek scientific world to recent findings about deterministic chaos, bafflements are a usual way, according to which old paradigms are broken and mathematics and its knowledge(s) go on in their development. In this way new, mathematical knowledge is often generated through conceptual and cognitive discontinuities: they challenge mathematical sense-making for students and for those knowledgeable of the discipline, generally creating what many authors call *conflicts* [69, 75, 76], others *contradiction*—which refers to the "accumulating structural tensions within and between activity systems" ([27], p. 137)—, and others *obstacles* [19], often accompanying the substantive with adjectives, like cognitive, epistemological, etc., according to their main focus, on students' or teachers' processes or on the discipline content. For example, in Brousseau, the notion of epistemological obstacle is introduced as follows:

> Obstacles of really epistemological origin are those from which one neither can nor should escape, because of their formative rôle in the knowledge being sought. They can be found in the history of the concepts themselves ([19], p. 87).

Hence, an obstacle is "a piece of knowledge or a conception, not a difficulty or a lack of knowledge" (*ibid*., p. 99). As such, an obstacle can be revealed by learners' errors, but it must not be confused with errors, which are the effect of the obstacle, namely "of a previous piece of knowledge which was interesting and successful, but which now is revealed as false or simply unadapted" (*ibid*., p. 82). However, an obstacle has also a cognitive nature, insofar it entails the necessity of a fresh way of thinking that the new knowledge would require but apparently is not coherent with the previous one and encounters difficulties to be activated. Difficulties in an obstacle may be particularly subtle since, at first glance, the relationship between the old and the new knowledge seems a contradiction between the two, but generally it is not so: the new frame simply enlarges the old one putting forward a new standpoint, which allows to embrace the previous one in a new setting, which does contradict the older one in case the older framework is still used. Sfard [68] thus defines what she names *commognitive conflict* as follows:

> a situation in which communication is hindered by the fact that different discursants are acting according to different meta-rules (and thus possibly using the same words in differing ways). Usually, the differences in meta-rules that are the source of the conflict find their explicit, most salient expression in the fact that different participants endorse contradicting narratives ([68], p. 374).

Because the same words are used in different discourses, incommensurability may be invisible to discourse users. Instead, they may perceive an apparent *incompatibility* of narratives. But these narratives are not talking about the same thing. For example, Euclidean Geometry is incompatible with the Hyperbolic one, but it is only incommensurable with the geometry necessary to describe "strange" objects like the Sierpiński triangle [4].

Thus, the term *conflict* indicates a piece of new knowledge added to the old one, when this reveals inadequate to solve a fresh problem (epistemological side); the new knowledge is incommensurable but not contradictory with the old one and consequently asks for new ways of reasoning (cognitive side).

Sfard ([68], p. 574), however, specifies that "the notion of commognitive conflict should not be confused with the acquisitionist idea of cognitive conflict, central to the well-known, well-developed theory of conceptual change [68]". She lists three reasons for this distinction:

- the first within the locus of the conflict, that is, by contrasting the truth-falsity of a concept—of which the world is the arbiter—, with the idea of incommensurability between discourses;
- the second is in their significance for learning, that is, from "an optional pedagogical move, particularly useful when students display 'misconceptions'" ([68], p. 575), to an indispensable source of metalevel mathematical learning;
- the third in the way the conflict is to be resolved, that is, moving from the principles of incompatibility and noncontradiction—two supposed contradictory narratives are also mutually exclusive, with a common criterion to reject or endorse and label one as true—, to a conflict resolution as making sense of other people's thinking-talking about the world with "a gradual acceptance, 'customization', and rationalization—figuring out the inner logic—of other people's discourses" ([68], p. 576).

It is from these reflections that the first outcome of my research work emerges: the introduction of the concept of *cultural conflict*. Although it is not a novel terminology, its application to the field of research in Mathematics Education is highlighted.

It is not a matter of hierarchical dynamic in which there's the "main" culture and then the "other" culture, and conflict results almost always in the logic of difference being determined by the main culture. The "other" is not the lesser, the less cultured, the less civilised. By encouraging people (students and teachers, but also researchers and scholars) to move into a different cultural paradigm, people can gain awareness of what Jullien [41] defines as *unthoughts*, i.e. those aspects that escape people's consciousness when they are immersed in their own culture. It is not so much a matter of understanding foreign cultures, thought of as homogeneous spheres with marked boundaries, but of an "interaction with foreignness" (Welsch, 1999, in [14]). Meeting foreign cultures leads to looking at one's own practices. The concept of cultural conflict therefore extends the idea of conflict presented above, but with a change of context: here it is about teachers' professional development and not about students' learning. Ramploud et al. [62] in this context call the "experience of" or "moment of *crisis*"—the main assumption of the teachers' professional development process through Cultural Transposition—what others call *obstacle*, *contradiction,* or *conflict*.

This "cultural" extension embeds the concept of "conflict" within the theoretical framework of the semiosphere. Table 1 describes the differences between the concepts of cognitive, commognitive and cultural conflict, and provides the definition.

According to these theoretical elements, my research questions are: How can the networking of the Semiosphere lens within the research stream of mathematics teacher education contribute in a sustainable and meaningful way to the transformation of the discourse and practices of mathematics teachers in Italy? How does incorporating cultural and social aspects into mathematics teacher education impact the outcomes of professional development programs for Italian mathematics teachers?

Table 1 Comparison of conflict concepts (adapted from [68], p. 756)

Concept	Cognitive conflict	Commognitive conflict	Cultural conflict
Ontology: The conflict is between	The interlocutor and the world	Incommensurable discourses	Asymmetrical contexts within discourses
Role in learning	Is an optional way for removing misconceptions	Is practically indispensable for metalevel learning	Is a transformative process through a redefinition of the problem and a critical self-reflection of the assumptions
How is it resolved?	By student's rational effort	By student's acceptance and rationalization (individualization) of the discursive ways of an expert interlocutor	By the teacher's professionalism but without betraying the context That is, with an explosion of significance: by making explicit learner's *unthought* [but potentially activated by the encounter with foreignness]

4 Methodology

The study presents the analysis of two case studies, henceforth identified as LS1 and LS2, achieved through Lesson Study and conducted with 5th and 12th grade students in the areas of probability and combinatorics respectively. These cases represent a subset of a more extensive collection of experiences, comprising a total of 39 Lesson Study cycles, realised during the years 2018–2021 [49]. I collected audio and video recordings of various meetings of the Lesson Studies, with written documentation, such as Lesson Plans, teachers' notes, and students' writings. Then, each meeting was verbatim transcribed, integrated with video-screenshots and images, and merged with the other sources for a holistic analysis.

Data analysis has been carried out following Lotman's semiotic analysis based on his spatial model (Fig. 2):

a. First and foremost, all the collected data are considered as texts,[3] that is, constituted by "webs of significances" to be interpreted. As such, the texts allow the identification of the moments of movement of the Semiosphere. This is a phase of interpretation that is never definitive and always dynamic. Two interpreters (the author and a facilitator) independently approached the texts describing the *comprehensiveness* and *dimension* of the Semiosphere of each Lesson Study cycle. I identify what is inside (known—"IN" in Fig. 2), and what is outside (to make sense of—"ES" in Fig. 2) from the semiosphere.
b. The *orientation* movements of signification are categorised in outward push or inward reception (left and right images respectively, in Fig. 2). I distinguish if the discourses are intentionally directed to encounter the Other or to critically observe the inner self's way of conceiving education.

Once it is described, the dialogical relations/webs of the Semiosphere, i.e. the asymmetries are manifest. The cultural conflict is described through a narrative of the interpreter who

[3] A text is an assemblage of signs, as conventionally used in Semiotics. See Semiotics for Beginners by Daniel Chandler http://visual-memory.co.uk/daniel/Documents/S4B/.

detected and described it from the sample of those discourses with asymmetry. Then, the observed oriented movements were interpreted.

I schematically summarise the described methodological process; the labelling of the intermediate phases follows Lotman's modelling described in the theoretical framework.

Initial phase) *data collection and transcription phase*. The data themselves lead to the development of analysis tools.

Intermediate phase) *research and description phase*. The data are considered as texts; the interpreter/researcher looks for moments of conflict, describing the space (known or to make sense of) and identifying asymmetries.

Final phase) *categorisation phase*. The texts (conflicts in teachers' discourse identified in the intermediate phase) are categorised into "directed to encounter the Other" or "directed to critically observe the inner". So, finally, the learner's action is described: how the teacher's professionalism enables them to cope with the unthought.

In this paper, for space reasons, I focus on only two convenient samples of asymmetrical discourses (LS1 and LS2). They are excerpts of transcriptions of the post-implementation discussion meeting of two different Lesson Studies. To have a sample of two dissimilar universal spaces, LS1 is part of an experience in Primary School, while LS2 is in Secondary School. As case studies, they are not exhaustive of the most robust findings of the study: in fact, here for contingency, only two cases of inward reception are illustrated. However, some outward push scenarios also occur, which we refer to in Manolino [49]—although the largest percentage of orientation movements proved to be inward reception, as here presented.

The confidentiality of participants is maintained through the comprehensive pseudonymisation of data.

5 Data analysis

5.1 LS1: Tombola game to teach probability and fractions

Initial phase) In a grade 5 class in a Northern Italy school, students are asked to play a simplified version of the Italian tombola,[4] with a 15-number card per team on an entire board of the first 50 natural numbers instead of the usual 90. There are 20 students in the class, 11 boys and 9 girls. The teacher divides the class into 4 teams (blue, green, red and purple) with 5 children each, giving each team member the same card. The teacher stops the game when one team is missing only one number. A question is posed to the class: What is the probability that the missing number will not be drawn?

This is the first time the children have experienced the concept of probability and they have never faced the concept of fraction. This lack is due to a strong social malaise in the class, generated by the distant learning of the pandemic situation.

The blue team is now only missing the number 32. The children spontaneously understand that in order to know which numbers are still in the bag, all they have to do is check the missing numbers on the total board. Numbers 16–17–25–27–32–37–44–46–49 are still in the bag.

Figure 3 shows the answers of three children. I precisely present these answers because they are the ones in which the children use the number 1 (corresponding to the possibility of drawing 32) as the "denominator".

[4] https://en.wikipedia.org/wiki/Tombola_(raffle) A well-known game in Italy, from children to the elderly. In order to win one of the prize lots, the player must be the first to complete it, i.e. the first to see all the numbers in his sequence come up.

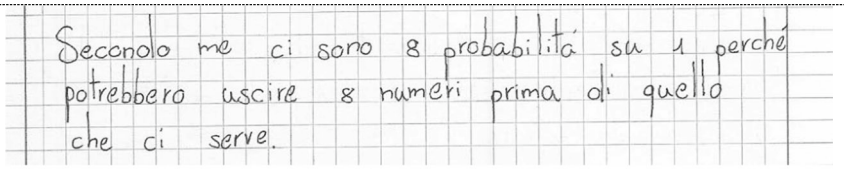

Fig. 3a. Emilia's answer to question 3.

[Text translation: In my opinion, there are 8 probabilities out of [over] 1, because 8 numbers could come up before the one we need.]

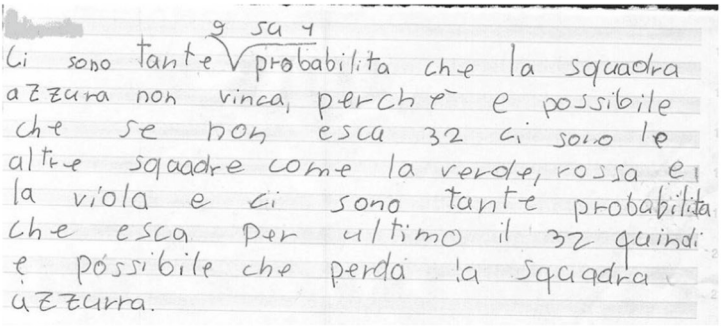

Fig. 3b. Ermanno's answer to question 3.

[Text translation: There is a high (9 out of [over] 1) probability that the blue team does not win, because it is possible that if 32 does not come out there are other teams such as the green, red and purple and there is a high probability that 32 comes out last so the blue team may lose.]

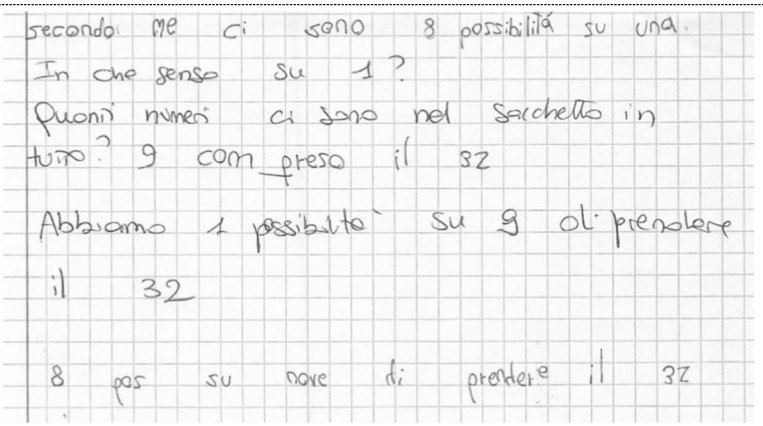

Fig. 3c. Daniele's answer to question 3.

[Text translation: In my opinion, there are 8 possibilities out of [over] 1. *What do you mean out of 1? How many numbers are in the bag in total?* 9 including 32. *We have 1 possibility out of 9 to get 32.* 8 possibilities out of 9 to [not] get 32. (In italics teacher's words when interacting with the student as "hand-lenders", i.e. transcribing part of the discourse of children with mathematics learning disabilities.)]

Fig. 3 The three children's answers with translations

This happens within the first Lesson Study cycle of two, implemented in December 2020. Below there is an excerpt of the teachers' discourse from the post-implementation discussion meeting.

Silvia: It seems to me that Emilia's writing is just a matter of how she expresses her thought.
Vanessa: I don't know. Knowing Emilia's way of thinking, I don't know if it was only the way she expressed her thoughts.
[...]
Chiara: In my opinion, Emilia does not manage to grasp the number of possible cases, that is, she only looks at the favourable cases. The favourable are 8. She looks at the possibilities, the numerators. And they are 8. And then the "winning" possibility [from the first question] is 1. Thus, for her, it is not 8 out of 1 in the sense of probability. […] there is not really the 9. It is not something considered. For her, there are only those who win or those who lose, there is no whole situation. Here, perhaps, the work on the idea of fraction, of the part-whole relation, is important.
[...]
Noemi: In my opinion, Emilia is paradigmatic, from the point of view of the passage from expressing one's thought. I also believe that it's just a matter of expressing thought. Unfortunately, we are in the passage from the expression of thought to a mathematical language. She is paradigmatic in messing up the 8 out of 1, which is a mathematical expression that she doesn't dominate. She makes a short circuit between the language to express the mathematical thought, that is a mathematical language, and what is her verbal, narrative thought. This is a knot that hinders people like me in mathematics, for example. To understand I always have to tell myself the whole story, but it's a mess when you get to a certain level [of the mathematical discourse]. They have a common logical origin, but the mathematical one is a different kind of thinking, that [can fail] if you don't build on solid foundations…
[…after a while in teachers' discussion…]
Enzo: In this experience, there was a relationship between three aspects: one is the very concrete experience, the game, [...] then there is the request that the teacher's request for them to express a thought[...], and then there is a third aspect that is the number, the mathematics. Vanessa, you demanded yourself if you had done the right thing and if it hadn't been misleading for some children to say: "Now let's do maths" […]. Mathematics in this case is not doing operations in columns, it is not a technique or an algorithm, but it is linked to reasoning. To reinforce that mathematics is, first of all, thought and reasoning before numbers, operations, etc... in my opinion is very important and you did a very good thing. It is completely normal if some children are more puzzled, they are facing their puzzlement. […] However, in this specific situation, I asked myself some questions, […]. In particular in the passage from thinking and reasoning to mathematics, to numbers, to fractions. [...] it would be interesting to see how the children would have worked from the point of view of the transition from thinking to mathematics if they had had more experience of fractions and had known this thing in other areas and had been able to transfer it to this level.[…] Another aspect [...][is] the fact of reading their texts, their thoughts[...], because it is not only useful for the child, but also useful [...] for us as teachers, because then you have a variety of thoughts that probably also at the oral level are not expressed or are expressed in other ways. […] it is really important to document the things that are done. Not all of them, but for some things it is really important to keep the documentation, because

> then some things escape us with the passing of time, especially the changes that occur in children['s discourse].

Intermediate phase) The spatial model sees the episode as a text of the Semiosphere whose centre is an Italian primary school children's sense of number and mathematical reasoning: "*Emilia's writing is just a matter of how she expressed her thought*", but also the teacher-student relationship within a mathematical classroom: "*She makes a short circuit between the language to express the mathematical thought, that is a mathematical language, and what is her verbal, narrative thought. This is a knot that hinders people like me in mathematics*." The external is the time constraints of a Lesson Study, the requirement to document the experience: "*it is really important to document the things* […]*, because then some things escape us with the passing of time, especially the changes that occur in children[']s discourse]*", and the theory on mathematical discussion [10]. The Lesson Study methodology, with the constraint of having the teachers sit at the same table talking about didactics, allowed the teachers to get involved in a field that was hostile to them: the probability. This challenged the teaching practices of fractions, but also classroom management: "*it would be interesting to see how the children would have worked from the point of view of the transition from thinking to mathematics if they had had more experience of fractions*"; "*if it hadn't been misleading for some children to say: "now let's do maths"[…]. Mathematics in this case is not doing operations in columns, it is not a technique, an algorithm, but it is linked to reasoning*". The conflict occurs between the fatigue expressed by teachers of not always having the opportunity to reflect on the details of a student's production, to fully understand its meaning: "*I don't know*"; "*some things escape us with the passing of time*", and the opportunity to do so either during Lesson Study or by activating new classroom practices (mathematical discussion, collecting documents).
Final phase) Here I identify a moment of movement of inward reception (Fig. 2—right). By encountering the crucial role of documentation, teachers realised what they missed: allowing themselves time to enter the children's discourses. Understanding children's thoughts, deep down and with didactic continuity.

5.2 LS2: The Alpinists' Problem as an introductory combinatorics lesson

Initial phase) This study took place in a secondary school in Italy during the 2020/21 school year. The LS group consisted of 5 teachers with 10–20 years of teaching experience. Multiple cycles of LS were implemented, the extract below is from the third cycle and involved a grade-12 class. The teachers' research problem I am considering (for further details see [3]) was to investigate students' combinatorial reasoning. It was decided to plan one lesson for the introduction to combinatorics. The decision to focus on combinatorics was made because it is a suitable topic for both grade levels and the didactics of combinatorics presents a challenge for both students and teachers [15]. The focus of the lesson was a problem-based group activity followed by a whole class discussion. The LS group's challenge was to select a problem that was easy to understand but challenging enough to engage and stimulate all students during an introductory combinatorics lesson. In the end, the teachers opted for the Alpinists' Problem ([64], p. 309), a suitable choice due to its simple formulation:

> 8 alpinists need to rope up in line to walk across a glacier. Two of them are beginners and cannot be placed neither in the first nor in the last position. How many ways are there to form the line?

The students were presented with the problem and were expected to collaborate in groups, without any assistance from the teachers, and then to discuss their solutions and strategies

in a mathematical discussion [10]. Below there is an excerpt of the teachers' discourse from the post-implementation discussion meeting of the third cycle.

> **Tiziana**: At a certain point, in the problem but also in the final solution, it was said that you can move the letter to a different position, or you can fix the position and choose a letter to put in. Because it is different…
> **Alessio**: Yes, you can see it in two ways.
> **Tiziana**: Actually, also in the final solution. You [Ludovica] had summed up the reasoning in a different way compared to what the student said. Writing 6x5, the student didn't say: 'For the first and the last place I have 6 and 5 possibilities'. He said: 'Beginners can stay in 6 places. I take the first, he can stay in one of the six central places'.
> […]
> Then one takes the second beginner and can put him in 5 places. So he [the student] took the places and put in them the alpinists.
> […]
> The student said this, and then you [Ludovica] summed up in a different way that was the way we solved the problem [when preparing the LP].
> […]
> **Ludovica**: Often, when a person has an idea, s/he [referring to the generic person] doesn't listen carefully to the solution, and doesn't have time to understand deeply. I didn't get this solution, you [Tiziana] are telling me now. I saw the right calculation, the student was explaining and I summed up to the class in the way I understood it. This thing is really interesting.

Then, the teachers also discussed further strategies proposed by other groups of students, such as the solution by subtraction, i.e. 8! - 2×7! - 2×7! + 2×6! or 8! - 26×6!.

> **Vittoria**: I really appreciate your explanation about the solution by subtraction, 8! minus… I would not have been able to do it. And probably, for my idea, I wouldn't have insisted on this kind of solution, I would have insisted on the other way. I would have dropped the solution by subtraction, pretending not to see it. And probably this is wrong, because we don't all have the same ways of reasoning, and so it is not respectful of the students' ways of working.

Intermediate phase) The spatial model sees the episode as a text of the Semiosphere whose centre is teachers' habit of conducting a mathematical discussion. It is only during the LS group discussion that teachers became aware of a number of different ways of solving the problem. The act of summing up students' strategies becomes a locus of asymmetry.
During the Lesson Plan design, teachers' solution of the problem followed this reasoning: you have 8 places for the 8 alpinists. In the first place you can choose among 6 experts, in the last place you can choose among 5 experts, and then you have to fill the 6 central places with the remaining 6 alpinists. In formula: $D_{6,2} \times P_6 = 6 \times 5 \times 6!$. During the lessons, many other strategies were found by the students. For example, in the transcript, Tiziana observed that the solution proposed by a group had the same calculation but a different reasoning: in the first solution (the teachers' one) the objects are the alpinists, in the second the objects are the positions in the rope. It is during the LS group discussion that Ludovica, who delivered the lesson, became aware of the different strategies proposed by the student during the lesson, and of the fact that her way of acting has modified students' reasoning: "*I didn't get this solution, you are telling me now. I saw the right calculation, the student was explaining, and I summed up for the class in the way I understood it*".

The cultural conflict is in the asymmetrical context generated by the encounter between, on the one hand, the teachers' usual practices and their spontaneity in solving a simple task during the planning phase, and on the other hand, the foresight during the planning phase of students' possible answers and the mutual peer observation and the LS group discussion. When planning the resolution of a problem, secondary school teachers often limit themselves to finding one solution, or sometimes not even that, correctly relying on their high level of knowledge in mathematics. However, the demand for a thin planning, as required by the Lesson Plan, shifts the focus from the single problem solution to the mathematical content itself: teachers here become able to recognise different strategies and argumentations supporting the same written calculation—"*This thing is really interesting*". Vittoria also reports the fact that, observing the lesson delivered by another teacher, she realised that in the same situation she would have acted differently: "*I wouldn't have insisted on this kind of solution. I would have insisted on the other way. I would have dropped the solution by subtraction, pretending not to see it*". The cultural conflict enhanced Vittoria the possibility to envision other ways of acting: "*probably this is wrong, because we don't all have the same ways of reasoning*", in contrast to her "*probably, for my idea*". "Her *idea*" does not change. She still considers the strategy by subtraction to be less suitable for solving the problem itself, however, the asymmetry of the context leads her to reflect on why to propose this problem: not so much to get all the students to solve it, but to reason together about different ways of solving, different combinatorial reasoning (the mathematical content).

Final phase) Here I identify a moment of movement of inward reception (Fig. 2—right). By encountering the collaborative sequence planning-implementation-observation-discussion, teachers rethought and improved their awareness of combinatorial reasoning. They become aware of the wide variety of strategies that students use in solving the problem, and this enables them to effectively lead the whole class discussion not only on the problem but on its mathematical content.

6 Discussion and conclusions

I assumed two premises: (1) that teaching is a culturally situated activity and (2) that, in accordance with Sfard's [69] framework, learning is the transformation of discourses. Looking then at teachers' discourses as a window on teacher learning (e.g., [24]), and to address my research questions, I modify Lotman's theoretical framework, which was originally developed for semiotics research, for use in the field of Mathematics Education, and more specifically, for teacher education. Considering discourses as texts, the analysis was semiotic. I identified cultural asymmetries and conflicts in two case studies, LS1 at primary school and LS2 at secondary school. They have been characterized as two inward receptions, Semiosphere's movements "directed to critically observe the inner". The findings suggest that the Lesson Study, when embedded in a framework of Cultural Transposition, supports the transformation of teachers' discourses.

By encountering the crucial role of documentation in LS, teachers realised that they missed allowing themselves time to enter the students' discourses. By encountering the LS collaborative sequence planning-implementation-observation-discussion, teachers enable themselves to effectively lead the whole class discussion not only on the contextual problem, but on its mathematical content.

Within a novel ontological approach—i.e., a cultural-semiotic and holistic approach, delving beyond cognitive aspects to encompass metacognitive, affective, and relational

dimensions—, the outcomes of this study are both a response to the theoretical need for research in the field of Mathematics Education and a practical proposal for teacher education. It is therefore a conscious culturally sensitive, research-driven teacher education that succeeds in being the hub of professional development. Indeed, since semiosis is unlimited—as in the Peircean view [55], but here extended to the semiosphere and embedded in a mathematics education context—the process of interpreting the significance of practices can never end. The asymmetry pushes teachers to elaborate their own transposition processes differently from school to school and sometimes also within the same school but in successive different steps. What is generated is an unending process of semiosis. This should not demoralise us, but rather keep high our attention because everything can always be a source of new consciousness.

Of course, it is just the beginning. There are many possible future developments for this study: for instance, it could also be explored whether the Semiosphere could be a useful theoretical lens to describe disability and inclusive aspects. Inclusion is indeed a characteristic aspect of our curriculum, and the Italian inclusive school is a *unicum* in the European context, but fraught with problems. Defining what is meant by disability is one of the greatest cultural challenges of our time.

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Declarations

Conflict of interest The corresponding author states that there is no conflict of interest.

Data availability Data supporting this study are not publicly available due to GDPR. Access to the data is subject to a data sharing agreement.

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