



PROCEEDINGS
OF THE
46th CONFERENCE
of the International Group for the
Psychology of Mathematics Education

Haifa, Israel | July 16 – 21, 2023



PME-46
MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

Volume 3

Editors: Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel and Michal Tabach



PROCEEDINGS
of the
46th CONFERENCE
of the International Group for the
Psychology of Mathematics Education

Haifa, Israel

July 16 – 21, 2023

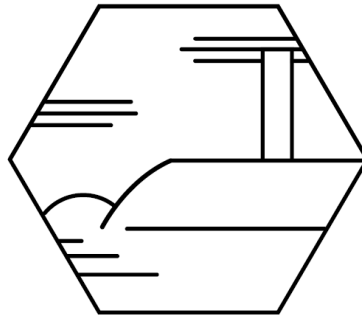
Editors:

Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel and Michal Tabach

Volume 3

Research Reports

H – O



PME-46
MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

Cite as:

Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel & Michal Tabach (Eds.). (2023)
Proceeding of the 46th Conference of the International Group for the Psychology of
Mathematics Education (Vol. 3). University of Haifa, Israel: PME

Website: <https://pme46.edu.haifa.ac.il/>

The proceedings are also available via

<https://www.igpme.org/publications/current-proceedings/>

Copyright @ 2023 left to the authors

All rights reserved

ISSN: 0771-100X

ISBN: 978-965-93112-3-1

Printed by the University of Haifa

Logo Design: Sharon Spitz (<https://www.sharonspitz.com/>)

Cover design: Sharona Gil (Faculty of Education, University of Haifa)

TABLE OF CONTENTS VOLUME 3

RESEARCH REPORTS (H-O)

FACILITATING LEARNERS' APPRECIATION OF THE AESTHETIC QUALITIES OF FORMAL PROOFS: A CASE STUDY ON A PAIR OF JUNIOR HIGH SCHOOL STUDENTS	3-3
Hayato Hanazono	
CO-LEARNING THE DIFFERENCE MEANING FOR MORE-THAN SITUATIONS WITH/FROM A STRUGGLING STUDENT	3-11
Cody Harrington, Ron Tzur, Emine B. Dagli, Dennis DeBay, and Megan Morin	
THE ROLES PRESERVICE TEACHERS ADOPT IN MODELLING-RELATED PROBLEM POSING	3-19
Luisa-Marie Hartmann, Stanislaw Schukajlow, Mogens Niss and Uffe Thomas Jankvist	
THE COMPLEXITY OF GRAMMAR IN STUDENTS' TALK: VARIATIONS IN EXPRESSING FUNCTIONAL RELATIONSHIPS BETWEEN TWO QUANTITIES	3-27
Kerstin Hein and Katharina Zentgraf	
MATHEMATICS TEACHER EDUCATORS IN AN UNKNOWABLE WORLD: TEACHING MATHEMATICS FOR CLIMATE JUSTICE	3-35
Tracy Helliwell and Gil Schwarts	
EXPLORING DEVELOPING PATTERNS OF MATHEMATICAL IDENTITY WORK BY GIVING ATTENTION TO EMOTIONAL HUE AND TONE OF VOICE IN THE ACT STORYTELLING	3-43
Rachel Helme	
STUDENT BEHAVIOR WHILE ENGAGED WITH FEEDBACK-ENHANCED DIGITAL SORTING TASKS	3-51
Arnon Hershkovitz, Michal Tabach, Norbert Noster and Hans-Stefan Siller	
COMPARING STUDENT VALUES AND WELLBEING ACROSS MATHEMATICS AND SCIENCE EDUCATION	3-59
Julia L. Hill, Margaret L. Kern, Wee Tiong Seah and Jan van Driel	
THE CONNECTION BETWEEN MATHEMATICS AND OTHER FIELDS: THE DISCIPLINE OF MATHEMATICS VS. MATHEMATICS EDUCATION	3-67
Anna Hoffmann and Ruhama Even	

COMPARING TEACHER GOALS FOR STUDENT FOCUSING AND NOTICING WITH STUDENT OUTCOMES FOR FOCUSING AND NOTICING	3-75
Charles Hohensee, Sara Gartland, Yue Ma and Srujana Acharya	
WHY MANY CHILDREN PERSIST WITH COUNTING	3-83
Sarah Hopkins, James Russo and Janette Bobis	
INFLUENCE OF FIELD-DEPENDENCE-INDEPENDENCE AND SYMMETRY ON GEOMETRY PROBLEM SOLVING: AN ERP STUDY	3-91
Hui-Yu Hsu, Ilana Waisman and Roza Leikin	
CULTURAL VARIATIONS IN THE QUALITY AND QUANTITY OF STUDENTS' OPPORTUNITIES TO PARTICIPATE IN CLASSROOM DISCOURSE	3-99
Jenni Ingram	
SNAPSHOTS OF CURRICULAR NOTICING: PLANNING A SUBTRACTION ALGORITHM LESSON IN PRIMARY EDUCATION	3-107
Pedro Ivars and Ceneida Fernández	
THE DEVELOPMENT OF CONCEPTIONS OF FUNCTION - A QUALITATIVE LONGITUDINAL STUDY ON THE TRANSITION FROM SCHOOL TO UNIVERSITY	3-115
Tomma Jetses	
LEARNING ABOUT STUDENT'S STRATEGIES BASED ON AUTOMATED ANALYSIS: THE CASE OF FRACTIONS	3-123
Amal Kadan-Tabaja and Michal Yerushalmy	
HOW A TEACHER'S PROFESSIONAL IDENTITY SHAPES PRACTICE: A CASE STUDY IN UNIVERSITY MATHEMATICS	3-131
Thomais Karavi	
INETWORKING THE VARIATION THEORETICAL PRINCIPLES IN A PROBLEM-SOLVING BASED MATHEMATICS INSTRUCTION TASK DESIGN STUDY	3-139
Berie Getie Kassa and Liping Ding	
MATHEMATICAL PROVING FOR SUBVERSIVE CRITICAL THINKING	3-147
Elena Kazakevich and Nadav Marco	
STRATEGIES FOR PROOF CONSTRUCTION (SELF-REPORTS VS PERFORMANCE) - IS PRIOR KNOWLEDGE IMPORTANT?	3-163
Katharina Kirsten and Silke Neuhaus-Eckhardt	

EFFECT OF REPRESENTATION FORMATS ON STUDENTS' SOLVING PROPORTION PROBLEMS	3-171
Tadayuki Kishimoto	
OPEN-ENDED TASKS WHICH ARE NOT COMPLETELY OPEN: CHALLENGES AND CREATIVITY	3-179
Sigal Klein and Roza Leikin	
THE DISCOVERY FUNCTION OF PROVING BY MATHEMATICAL INDUCTION	3-187
Kotaro Komatsu	
PRE-SERVICE TEACHER TRAINING WITH AI: USING CHATGPT DISCUSSIONS TO PRACTICE TEACHER-STUDENT DISCOURSE	3-195
Ulrich Kortenkamp and Christian Dohrmann	
TEACHING MATHEMATICS WITH TECHNOLOGIES: PROFILES OF TEACHER CHARACTERISTICS	3-195
Timo Kosiol and Stefan Ufer	
RELATIONSHIPS BETWEEN PROSPECTIVE TEACHERS' HEART RATE VARIATION NOTICING OF CHILDREN'S MATHEMATICS	3-211
Karl W. Kosko and Richard E. Ferdig	
IN-THE-MOMENT TEACHER DECISION MAKING AND EMOTIONS	3-219
Styliani-Kyriaki Kourti and Despina Potari	
ROCK'N'ROLL – EMERGENT AFFORDANCES AND ACTIONS DURING CHILDRENS' EXPLORATION OF TOUCHTIMES	3-227
Christina M. Krause and Sean Chorney	
JUDGEMENT ACCURACY: COMPARING OPEN REPORTS AND RATINGS AS INDICATORS OF DIAGNOSTIC COMPETENCE	3-235
Stephanie Kron, Daniel Sommerhoff, Christof Wecker and Stefan Ufer	
INTERACTIONAL FORCES IN MULTILINGUAL DISCOURSES – A TEACHERS' PERSPECTIVE ON LEARNERS' AGENCY	3-243
Taha Ertuğrul Kuzu	
DRAWING ON CULTURAL STRENGTHS FOR COLLECTIVE COLLABORATION	3-251
Generosa Leach, Viliami Latu and Roberta Hunter	
BUILDING BRIDGES: THE IMPORTANCE OF CONTINUOUS MAGNITUDES IN EARLY MATHEMATICS EDUCATION FROM TWO PERSPECTIVES.	3-259
Tali Leibovich-Raveh	

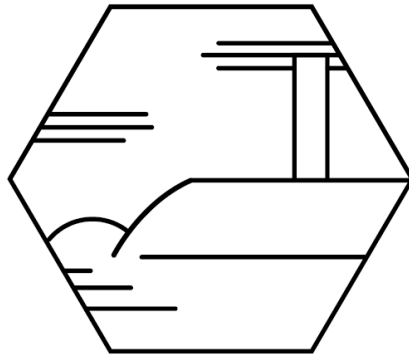
ENHANCING STUDENTS' CONCEPTUAL KNOWLEDGE OF FRACTIONS THROUGH LANGUAGE-RESPONSIVE INSTRUCTION. A FIELD TRIAL	3-267
Katja Lenz, Andreas Obersteiner and Gerald Wittmann	
ADULTS' AWARENESS OF CHILDREN'S ENGAGEMENT WITH GEOMETRICAL ACTIVITIES	3-275
Esther S. Levenson, Ruthi Barkai, Dina Tirosh, Pessia Tsamir, and Shahd Serhan	
A TEACHING INTERVENTION WITH DYNAMIC INTERACTIVE MEDIATORS TO FOSTER AN ALGEBRAIC DISCOURSE	3-283
Giulia Lisarelli, Bernardo Nannini and Chiara Bonadiman	
MORE THAN JUST THE BASIC DERIVATION FORMULA: THE IMPACT OF PRIOR KNOWLEDGE ON THE ACQUISITION OF KNOWLEDGE ABOUT THE CONCEPT OF DERIVATIVE	3-291
Kristin Litteck, Tobias Rolfes and Aiso Heinze	
SECONDARY-TERTIARY TRANSITION OF INTERNATIONAL STUDENTS: ONE STUDENT'S EFFORTS TO OVERCOME THE CHALLENGE OF LEARNING MATHEMATICS IN ENGLISH	3-299
Kim Locke, Igor' Kontorovich and Lisa Darragh	
SHIFTS IN LOCAL NARRATIVE IDENTITIES: A CASE OF LOW ACHIEVING STUDENTS.	3-299
Elena Macchioni	
ALGEBRAIC STRUCTURE SENSE IN A BLIND SUBJECT	3-307
Andrea Maffia, Carola Manolino and Elisa Miragliotta	
TEACHERS' LEARNING THROUGH ITERATIVE CONTEXT-BASED MATHEMATICAL PROBLEM POSING	3-315
Nadav Marco and Alik Palatnik	
TOWARDS THE NOTION OF CONCEPT GESTURE: EXAMINING A LECTURE ON SEQUENCES AND LIMITS	3-323
Ofer Marmur and David Pimm	
THE ROLE OF TOPOLOGY IN TWO-VARIABLE FUNCTION OPTIMIZATION	3-331
Rafael Martínez-Planell, María Trigueros and Vahid Borji	
REASONING AND LANGUAGE IN RESPONSES TO READING QUESTIONS IN A LINEAR ALGEBRA TEXTBOOK	3-339
Vilma Mesa, Thomas Judson and Amy Ksir	

DEVELOPING AN INTERNATIONAL LEXICON OF CLASSROOM INTERACTION	3-347
Carmel Mesiti, Michèle Artigue, Valeska Grau, and Jarmila Novotná	
MEASURING DATA-BASED MODELING SKILLS IN A COLLABORATIVE SETTING	3-355
Matthias Mohr and Stefan Ufer	
ANALYSIS OF HOW PRE-SERVICE MATHEMATICS TEACHERS INCLUDE SRL IN THEIR TEACHING PROPOSALS	3-355
Hidalgo Moncada, D., Díez-Palomar, J. and Vanegas, Y.	
LESSON STUDY AND IMPROVISATION: CAN TWO WALK TOGETHER, EXCEPT THEY BE AGREED?	3-371
Galit Nagari-Haddif, Ronnie Karsenty and Abraham Arcavi	
ATTENDING TO ARGUMENTATION: EXPLORING SIMILARITIES AND DIFFERENCES BETWEEN MATHEMATICS PRE-SERVICE AND IN-SERVICE SECONDARY TEACHERS	3-379
Samaher Nama, Maysa Hayeen-Halloun and Michal Ayalon	
A CARTESIAN GRAPH IS “A THING OF MOVEMENT”	3-387
Bernardo Nannini and Giulia Lisarelli	
INSTRUCTIONAL SHORT VIDEOS IN CALCULUS: THE MATHEMATICAL DIDACTICAL STRUCTURES AND WATCHING PATTERNS	3-395
Eli Netzer and Michal Tabach	
CONSTRUCTING A PROOF AFTER COMPREHENDING A SIMILAR PROOF – RELATION AND EXAMPLES	3-403
Silke Neuhaus-Eckhardt and Stefanie Rach	
THE ROLE OF TEACHERS’ PERSON CHARACTERISTICS FOR ASSESSING STUDENTS’ PROOF SKILLS	3-411
Michael Nickl; Daniel Sommerhoff, Elias Codreanu, Stefan Ufer and Tina Seidel	
ZPD NOTICING – A VIGNETTE-BASED STUDY INTO PRE-SERVICE TEACHERS’ ANALYSIS OF AN ALGEBRA CLASSROOM SITUATION	3-419
Yael Nurick, Sebastian Kuntze, Sigal-Hava Rotem, Marita Friesen, and Jens Krummenauer	
ON THE CONNECTION BETWEEN BASIC MENTAL MODELS AND THE UNDERSTANDING OF EQUATIONS	3-427
Reinhard Oldenburg and Hans-Georg Weigand	

MOTIVATIONAL AND EMOTIONAL ENGAGEMENT MEDIATES THE EFFECT OF FEATURES OF EDUCATIONAL TECHNOLOGY IN MATHEMATICS CLASSROOMS	3-435
Maria-Martine Oppmann and Frank Reinhold	
HOW DOES MATHEMATICAL KNOWLEDGE FOR UNDERGRADUATE TUTORING DEVELOP? ANALYSING WRITTEN REFLECTIONS OF NOVICE TUTORS	3-442
Tikva Ovadiya and Igor' Kontorovich	

VOLUME 3
RESEARCH REPORTS

H-O



PME-46
MATHEMATICS EDUCATION
FOR GLOBAL SUSTAINABILITY

ALGEBRAIC STRUCTURE SENSE IN A BLIND SUBJECT

Andrea Maffia¹, Carola Manolino² and Elisa Miragliotta³

¹University of Bologna, Italy; ²University of Valle d'Aosta, Italy;

³University of Pavia, Italy

Literature about the way visually impaired students approach mathematics is still very scarce, especially in the case of algebra, despite the fact that mathematical content is known to be increasingly accessible thanks to assistive technologies. This report presents a case study aimed at describing the process of algebraic symbols manipulation by a blind subject. Results show how the hearing can substitute the eye in the development of structure sense. Data analysis reveals that screen readers and coding languages (as LaTeX) have interesting potentialities in the development of structure sense for blind students.

INTRODUCTION

Several authors in the field of mathematics education have been discussing about the importance of visualization within different mathematical activities (e.g., Presmeg, 2006). The term ‘visualization’ itself refers to the sense of sight that may result as a way of accessing mathematical representations – including geometrical shapes, graphs, and formulas. According to Sfard (2008), the mathematical discourse itself is defined (among other features) by specific *visual mediators*, defined as “visible objects that are operated upon as a part of the process of communication” (p. 133). Radford (2010) describes the process of learning mathematics – particularly in the algebraic context – as a process of domestication of the eye, meaning “a lengthy process in the course of which we come to see and recognize things according to ‘efficient’ cultural means” (p.10). However, for some students, the sense of sight is not, totally or partially, suitable to access mathematical content because of visual impairments or complete blindness.

The literature about the learning of mathematics for blind students is very scarce, but we know that they can access mathematical contents through other senses (e.g., Alajarmeh et al., 2011; Healy & Fernandes, 2011). Indeed, Radford (2010) recognizes that the same lengthy process of domestication could happen for other senses, and Sfard (2008) defines *realizations* of mathematical objects as “perceptually accessible things” (p. 154), without specifying the nature of such perception. Furthermore, she recognizes that gestures can realize mathematical objects (Sfard, 2008). Healy and Fernandes (2011) have, indeed, observed that blind subjects may involve gestures in their appropriation of mathematical meanings; these authors see the gestures used by a blind subject as re-enactions of previously experienced activities. Their arguments are convincing in the case of geometrical figures and solids: the gesture may correspond to past experiences of touching, moving a finger along physical artifacts. Similarly, the description of a graph of a continuous function (or even a discrete graph) can be based on the embodied experience of motion (Núñez et al., 1999). On the contrary, other

mathematical representations – like algebraic symbols – appear much more detached from the sensorial experience and we wonder how blind students can get access to this kind of representations. Assistive technologies and the braille alphabet may be suitable means (Alajarmeh et al., 2011; Armano et al., 2018; Bouck et al., 2016).

To the best of our knowledge, there is no international literature in the field of mathematics education about algebraic symbol manipulation by blind students using screen readers. Hence, this study is a first step forward in filling this research gap by offering a *thick description* (Bell & Kissling, 2019) of the process of algebraic symbols manipulation performed by an experienced blind individual while solving an algebraic task.

THEORETICAL FRAMEWORK

The involvement of digital technologies within the process of teaching/learning mathematics has provided many new opportunities for visually impaired students who can rely on screen readers (and other assistive technologies) to access written text, including algebraic formulas (Alajarmeh et al., 2011; Armano et al., 2018). However, using algebraic symbols for mathematical problems solving does not only require reading the symbols, but being able to act upon them. While studies have focused on how digital textbooks can aid students' algebraic activity (e.g., Bouck et al., 2016), there is a dearth of research about systems for enabling students to act productively on symbols (Alajarmeh et al., 2011).

When we refer to algebraic symbol manipulation, we consider that it encompasses more than just the rote application of transformation rules. It involves a broader competence in using “equivalent structures of an expression flexibly and creatively” (Linchevsky & Livneh, 1999, p. 191), that is briefly named *structure sense*. According to Hoch and Dreyfus (2004), in the context of school algebra, structure sense can be described as composed of six abilities which are: (1) seeing an algebraic expression or sentence as an entity; (2) recognizing an algebraic expression or sentence as a previously met structure; (3) dividing an entity into sub-structures; (4) recognizing mutual connections between structures; (5) recognizing which manipulations it is possible to perform; (6) recognizing which manipulations it is useful to perform.

While describing a specific case study (see Method section), we are here interested in understanding if and how blind subjects can rely on their structure sense while solving an algebraic task, the accessibility of which is provided through digital tools. Our research question is: How can a blind subject rely on his structure sense while solving equations if supported by assistive technology?

METHODS

Due to the nature of our research question and because of the paucity of research literature on the topic, we have chosen to conduct an exploratory case study. Such design is recommended when the aim of the research is “to portray ‘what it is like’ to be in a particular situation, to catch the close up reality and ‘thick description’ [...] of

participants' lived experiences of [...] a situation" (Cohen et al., 2007). Aiming at describing how a subject draws upon his/her structure sense, we interviewed an adult person who has a strong education in mathematics, to whom we refer with the pseudonym of Antonio. Antonio, who has a degree in Physics and has worked as a fellow researcher for 2 years, became blind four years ago due to a degenerative pathology. He learned to use LaTeX with speech synthesis as a visually impaired undergraduate student, 10 years ago.

We opted for a task-based interview: the interviewee was asked to select two equations among those proposed by Hoch and Dreyfus (2004) and to solve them. Because of the space limit, we will present only some excerpts from the solution process of one equation (Figure 1b), about which the solver was particularly talkative. The task was presented through a PDF file (Figure 1) which was implemented with the *Axessibility* package for LaTeX (Armano et al., 2018). By adding a single line of code to the source LaTeX file (line 2, Figure 1a), this package automatically inserts a hidden alternative text in the PDF document at each formula which is then accessible to screen readers (e.g., Jaws, NVDA). On Antonio's computer the screen reader NVDA was installed, allowing him to hear the read-aloud of LaTeX code. The LaTeX code for the equation in focus is shown in the box 'a' of Figure 1. In particular, the command `\frac{}{}` represents a fraction having as numerator the content of the first curly braces and as denominator the content of the following curly braces.

Figure 1 consists of two rectangular boxes, labeled 'a' and 'b'. Box 'a' contains LaTeX source code with line numbers 1 through 5. Line 2 includes the `\usepackage[accsupp]{axessibility}` command. Line 4 shows a LaTeX equation: `\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})`. Box 'b' shows the compiled PDF output of the code in box 'a', displaying the mathematical equation: $\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$.

Figure 1. Example of LaTeX code (box a) and corresponding compiled PDF (box b).

Aiming at a thick description (Bell & Kissling, 2019) we collected several sources of data including “speech acts; non-verbal communication; descriptions in low-inference vocabulary; [...] recording of the time and timing of events; the observer's comments [...]; detailed contextual data” as prescribed by Cohen et al. (2007, p. 405). This was realized by recording the interview including in the audio- and video-recording of the interviewee (through a webcam) and capturing the interviewee's computer screen. The second author of this report acted as interviewer and took personal notes during the interview. The video was transcribed verbatim by the first author integrating the transcription of ‘what is said’ with descriptions of ‘what is done’ (e.g., Table 1) – as recommended by Sfard (2008) – and with screenshots. Screenshots have been elaborated adding arrows representing the movements of the cursor (Figure 2); the final position of the cursor is represented by a vertical line. The three authors have analyzed this enriched transcript by coding each line with the six components of structure sense

described in previous section. The coding process was discussed and reviewed among the three researchers until consensus was achieved.

RESULTS

The first excerpt refers to the first reading of the proposed equation. Antonio uses the functionalities of the Axessibility package (Armano et al., 2018) to hear the reading of the LaTeX code behind the PDF that we provided to him. Then, he decides to copy/paste the LaTeX code on the Microsoft Notepad application. The NVDA software reads the code out loud while his cursor navigates through the Notepad, as captured in the video.

Table 1. Enriched transcript of the first excerpt.

Line	What is said	What is done
1	Antonio: Let's see the structure. One fourth.	The cursor moves till the denominator of the first fraction and stops right before the closing curly brace.
2	Interviewer: While you are understanding the structure, would you like to tell it?	The cursor moves forward and stops after the minus sign.
3	Antonio: One fourth, yes.	The cursor moves back, before the minus sign.
4	Minus x over... Over x minus one. Minus x equals... Then there is the second member of the equation.	The cursor moves forward and stops right before the equal sign.
5	Five plus, open bracket. Then there is a bracket.	The cursor reaches the opening bracket.
6	One fourth inside the bracket.	The cursor moves back and forth over the <code>\frac</code> command.
7		The cursor reaches the end of the last fraction
8	Minus x over x minus one.	The cursor moves back and forth over the denominator of this fraction, then it goes back to the minus sign.
9	Closed bracket and that's it.	The cursor moves forward till the end of the equation.
10	Here I would start by working on the brackets.	

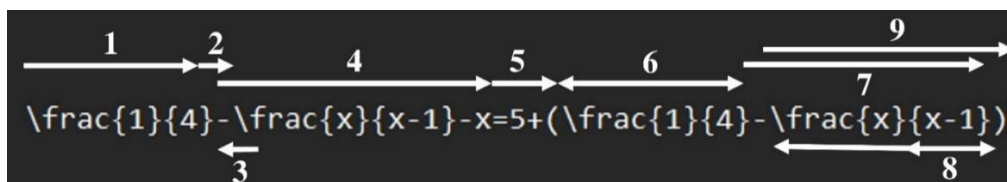


Figure 2. Cursor's movements during the first excerpt. Numbers are keyed to Table 1.

Despite not being familiar with the theoretical framework of this paper, Antonio starts by expressing the intent to understand ‘the structure’ (line 1). Since this strategy corresponds to reading the different sub-structures (the fractions, the parenthesis) of the equation, he is relying on the third component of structure sense, which is “divide an entity into sub-structures”. Indeed, by analyzing the movements of his cursor, we

can see that his attention is initially caught by the first fraction (lines 1-2). Then he analyzes the second fraction and stops when the left side of the equation finishes (line 4). The presence of the bracket is noticed (line 5) and then he goes back and forth over the two fractions within the brackets (lines 6-9). Then, we can notice that – even if the readers would normally force a left to right reading – Antonio analyses the structures of specific parts of the equation by realizing multiple readings of the identified sub-structures. This is particularly true for the fractions; their presence is highlighted by the LaTeX commands and the curly braces identifying the numerator and the denominator.

After this first excerpt, Antonio copies the whole equation on a second line in the Notepad. He recognizes that he can work inside the brackets first (component 5 of structure sense) and prepares the environment for doing so: he creates many blank spaces before the closing bracket (second line in Figure 3), then he states that he can calculate the least common multiple of the denominators (again component 5). He writes the denominator in the obtained blank space within braces (second line in Figure 3) and then adds a couple of braces before (third line) – so preparing the space for the numerator. He performs his calculations for the numerator between these braces (fourth and fifth line in Figure 3) and when he is done, he adds the command `\frac` before the braces (sixth line). Finally, he replaces the content of the brackets with the calculated fraction (last line in Figure 3). Then, he recognizes which manipulation he can realize and uses the braces as containers for organizing the structure of the result of such manipulations. After these manipulations, Antonio decides to work on the fractions on the first side of the equation, as shown in the excerpt in Table 2.

```

\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
)
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
{4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
){4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
{x-1-4x}{4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1}
{-1-3x}{4(x-1)})
\frac{1}{4}-\frac{x}{x-1}-x=5+(-\frac{3x+1}{4(x-1)})

```

Figure 3. Different phases of Antonio's manipulations of the fractions in brackets.

Table 2. Enriched transcript of the second excerpt.

Line	What is said	What is done
11	Antonio: Ok. Now let's see what was here.	The cursor moves back to the beginning of the equation.
12	There was one fourth.	The cursor moves till the end of the second fraction
13	And then there was the same thing as in the brackets, but outside.	Then the cursor moves back to the beginning of the equation.
14	Thus... Thus, the result is the same of the other side because it's equivalent.	The cursor moves till the end of the left side of the equation.
15	Then I can copy this.	The cursor moves to the fraction on the right side of the equation, which is then selected.

16		The fraction is copied in a new following line on the Notepad.
17	Then I copy ‘minus x equals’.	He types ‘-x=’.
18	Well, I can copy and paste the second member. Yes, I copy and paste it.	He selects, copies, and pastes the right side of the equation.
19	Thus, since they are... the fractions are equal but opposite in sign... because if... then, I bring at the second member what is in the first member, they are equal and opposite. The result should be...	The cursor goes back to the beginning of the equation and then moves till the end of the right side.
20	x equals minus five.	He moves to a new line and write ‘x=-5’.
21	Let me check...	The cursor goes back to the previous line and moves through the whole line, from left to right.
22	Yes, that should be the result.	

While reading the left side of the equation, Antonio recognizes the same structure of the expression within the brackets (line 13, component 2 of structure sense). Then, he understands that, instead of performing again all the manipulations shown in Figure 3, he can replace the fractions on the left side with the fraction calculated on the right side (lines 15-16, components 4 and 6), which was into the round brackets (Figure 1b). Having two identical sub-structures on the two sides of the equality, he decides to cancel them (lines 19-20, components 4 and 6). Hence, in this short excerpt we can notice many of the components of structure sense intervening and, considering the other parts of the transcript as well, we can observe all the six components of structure sense enacted.

DISCUSSION AND CONCLUSION

We can answer our research question by noticing that all the six components of structure sense have a role in Antonio’s solving of the equation using the Microsoft Notepad and NVDA reader for manipulating the equation represented in LaTeX code. In particular, Antonio uses the reader to read (and re-read) *self-selected* portions of the equation instead of simply reading from left to right. The notepad is used to manipulate the equation both within the same line (differently than what we are used to do with paper and pencil, Figure 3) or connecting different lines (e.g., lines 13-18).

We have noticed that during the first reading (Table 1) the equation is divided into sub-structures (component 3 of structure sense) and then Antonio recognizes which manipulations are possible (component 5) on these sub-structures. As observed for seeing subjects, brackets play a relevant role in structuring the equation (Hoch & Dreyfus, 2004). However, in this specific case, we can notice that also the use of the LaTeX code – allowed by the Axessibility package (Armano et al., 2018) – may play an important role in structuring the equation into sub-structures, since the `\frac` command is often a place where the cursor stops. Furthermore, the LaTeX code becomes not only a tool for reading mathematics, but a tool for doing mathematics as well (in the sense of Alajarmeh et al., 2011). This is visible when the curly braces are

used to organize the space of manipulation, distinguishing the numerator and the denominator of the algebraic fraction (Figure 3).

Antonio provides a telling example of how the LaTeX code can serve as a tool for symbolic manipulation with the interesting ‘side effect’ of being transformable in a PDF file which can be then read both by seeing students and visually impaired ones. As noted by Ahmetovic et al. (2021), LaTeX is a writing system used in all STEM disciplines, then its learning is both useful for academic achievement and for inclusivity. The study presented in this report suggests that the learning and use of LaTeX could promote and support structure sense especially for visually impaired students, but potentially not only.

We have also noticed that Antonio was able to recognize previously met (sub-) structures and use them to shortcut his manipulation (Table 1), so mobilizing many components (2-4-6) of structure sense. This recognition is realized after hearing the reading of the first part of the equation (line 13) by NVDA software; paraphrasing Radford’s (2010) words, Antonio’s ears have gone under a lengthy process of domestication through which they came to hear and recognize things according to an ‘efficient’ cultural mean. The reading of the equation and the memorized ‘sound-track’ acted as realizations of the equation in the sense of Sfard (2008) – being anything but visual. This fact corroborates that other senses than sight can successfully help not only in the rote manipulations of algebraic symbols, but in developing structure sense as well. This suggests that verbalization of the structure of algebraic expression may be an important step in the development of structure sense for blind people, but this is true for all the other students as well (Maffei & Mariotti, 2011). This observation strengthens what has been noted before in the case of LaTeX: adopting inclusive approaches to algebra teaching may be fruitful not only for impaired students, but for the whole class-group as well.

Surly, we must be cautious about the conclusion that we draw from a case study; in particular, we must consider that visual impairments are very different among them. For instance, Antonio was not completely blind during his high school studies and then he might rely on visual memories of algebraic expressions. Different results might be obtained in the case of students born blind and/or that are able to use Braille to read and write mathematical notations. Future developments of our research project will include subjects with different past histories about their disabilities and their learning of mathematics. Nevertheless, we hope that this report could offer a step forward in unveiling the (many) ways in which visual impaired solvers can successfully tackle algebraic equations.

REFERENCES

- Ahmetovic, D., Bernareggi, C., Bracco, M., Murru, N., Armano, T., & Capietto, A. (2021). LaTeX as an inclusive accessibility instrument for high school mathematical education. In S. Rodriguez Vazquez, T. Drake, D. Ahmetovic, & V. Yaneva (Eds.) *Proc. of the 18th Int. Web for All Conference* (pp. 1-9).

- Alajarmeh, N., Pontelli, E., & Son, T. (2011). From “reading” math to “doing” math: A new direction in non-visual math accessibility. In C. Stephanidis (Ed.), *Universal Access in Human-Computer Interaction. Applications and Services* (pp. 501-510). Springer Berlin.
- Armano, T., Capietto, A., Coriasco, S., Murru, N., Ruighi, A., Taranto, E. (2018). An Automatized Method Based on LaTeX for the Realization of Accessible PDF Documents Containing Formulae. In K. Miesenberger & G. Kouroupetroglou, (Eds.), *Computers Helping People with Special Needs* (pp.583-589). Springer.
- Bell, J. T., & Kissling, M. T. (2019). Thick description as pedagogical tool: Considering Bowers’ inspiration for our work. *Educational Studies*, 55(5), 531-547.
- Bouck, E. C., Weng, P.-L., & Satsangi, R. (2016). Digital versus Traditional: Secondary Students with Visual Impairments’ Perceptions of a Digital Algebra Textbook. *Journal of Visual Impairment & Blindness*, 110(1), 41-52.
- Cohen, L., Manion, L., & Morrison, K. (2007). *Research Methods in Education*. Routledge.
- Healy, L., & Fernandes, S. H. A. A. (2011). The role of gestures in the mathematical practices of those who do not see with their eyes. *Educational Studies in Mathematics*, 77(2), 157-174.
- Hoch, M., & Dreyfus, T. (2004). Structure sense in high school algebra: The effect of brackets. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 49-56). PME.
- Linchevski, L., & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, 40(2), 173-196.
- Maffei, L., & Mariotti, M.A. (2011). The role of discursive artefacts in making the structure of an algebraic expression emerge. In Pytlak, M., Rowland, T., & Swoboda, E. (Eds.) *Proc. 7th Conf. of the European society for Research in Mathematics Education* (pp. 511-520). ERME.
- Núñez, R. E., Edwards, L. D., & Filipe Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, 39(1), 45-65.
- Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. In A. Gutiérrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 205–235). Sense Publishers.
- Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For the Learning of Mathematics*, 30(2), 2-7.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge university press.