PROCEEDINGS

OF THE

46th CONFERENCE

of the International Group for the

Psychology of Mathematics Education

Haifa, Israel | July 16 – 21, 2023

PME-46

MATHEMATICS EDUCATION

FOR GLOBAL SUSTAINABILITY

Volume 3

Editors: Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel and Michal Tabach

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Research Reports

H – O

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Cite as:

Michal Ayalon, Boris Koichu, Roza Leikin, Laurie Rubel & Michal Tabach (Eds.). (2023) Proceeding of the 46th Conference of the International Group for the Psychology of Mathematics Education (Vol. 3). University of Haifa, Israel: PME

Website: https://pme46.edu.haifa.ac.il/

The proceedings are also available via

https://www.igpme.org/publications/current-proceedings/

ISSN: 0771-100X

ISBN: 978-965-93112-3-1

Printed by the University of Haifa

Logo Design: Sharon Spitz (https://www.sharonspitz.com/)

Cover design: Sharona Gil (Facuty of Education, University of Haifa)

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ALGEBRAIC STRUCTURE SENSE IN A BLIND SUBJECT

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Literature about the way visually impaired students approach mathematics is still very scarce, especially in the case of algebra, despite the fact that mathematical content is known to be increasingly accessible thanks to assistive technologies. This report presents a case study aimed at describing the process of algebraic symbols manipulation by a blind subject. Results show how the hearing can substitute the eye in the development of structure sense. Data analysis reveals that screen readers and coding languages (as LaTeX) have interesting potentialities in the development of structure sense for blind students.

INTRODUCTION

Several authors in the field of mathematics education have been discussing about the importance of visualization within different mathematical activities (e.g., Presmeg, 2006). The term 'visualization' itself refers to the sense of sight that may result as a way of accessing mathematical representations – including geometrical shapes, graphs, and formulas. According to Sfard (2008), the mathematical discourse itself is defined (among other features) by specific *visual mediators*, defined as "visible objects that are operated upon as a part of the process of communication" (p. 133). Radford (2010) describes the process of learning mathematics – particularly in the algebraic context – as a process of domestication of the eye, meaning "a lengthy process in the course of which we come to see and recognize things according to 'efficient' cultural means" (p.10). However, for some students, the sense of sight is not, totally or partially, suitable to access mathematical content because of visual impairments or complete blindness.

The literature about the learning of mathematics for blind students is very scarce, but we know that they can access mathematical contents through other senses (e.g., Alajarmeh et al., 2011; Healy & Fernandes, 2011). Indeed, Radford (2010) recognizes that the same lengthy process of domestication could happen for other senses, and Sfard (2008) defines *realizations* of mathematical objects as "perceptually accessible things" (p. 154), without specifying the nature of such perception. Furthermore, she recognizes that gestures can realize mathematical objects (Sfard, 2008). Healy and Fernandes (2011) have, indeed, observed that blind subjects may involve gestures in their appropriation of mathematical meanings; these authors see the gestures used by a blind subject as re-enactions of previously experienced activities. Their arguments are convincing in the case of geometrical figures and solids: the gesture may correspond to past experiences of touching, moving a finger along physical artifacts. Similarly, the description of a graph of a continuous function (or even a discrete graph) can be based on the embodied experience of motion (Núñez et al., 1999). On the contrary, other

2023. In M. Ayalon, B. Koichu, R. Leikin, L. Rubel & M. Tabach (Eds.). *Proceedings of the 46th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 307-314). PME 46.

mathematical representations – like algebraic symbols – appear much more detached from the sensorial experience and we wonder how blind students can get access to this kind of representations. Assistive technologies and the braille alphabet may be suitable means (Alajarmeh et al., 2011; Armano et al., 2018; Bouck et al., 2016).

To the best of our knowledge, there is no international literature in the field of mathematics education about algebraic symbol manipulation by blind students using screen readers. Hence, this study is a first step forward in filling this research gap by offering a *thick description* (Bell & Kissling, 2019) of the process of algebraic symbols manipulation performed by an experienced blind individual while solving an algebraic task.

THEORETICAL FRAMEWORK

The involvement of digital technologies within the process of teaching/learning mathematics has provided many new opportunities for visually impaired students who can rely on screen readers (and other assistive technologies) to access written text, including algebraic formulas (Alajarmeh et al., 2011; Armano et al., 2018). However, using algebraic symbols for mathematical problems solving does not only require reading the symbols, but being able to act upon them. While studies have focused on how digital textbooks can aid students' algebraic activity (e.g., Bouck et al., 2016), there is a dearth of research about systems for enabling students to act productively on symbols (Alajarmeh et al., 2011).

When we refer to algebraic symbol manipulation, we consider that it encompasses more than just the rote application of transformation rules. It involves a broader competence in using "equivalent structures of an expression flexibly and creatively" (Linchevsky & Livneh, 1999, p. 191), that is briefly named *structure sense*. According to Hoch and Dreyfus (2004), in the context of school algebra, structure sense can be described as composed of six abilities which are: (1) seeing an algebraic expression or sentence as an entity; (2) recognizing an algebraic expression or sentence as a previously met structure; (3) dividing an entity into sub-structures; (4) recognizing mutual connections between structures; (5) recognizing which manipulations it is possible to perform; (6) recognizing which manipulations it is useful to perform.

While describing a specific case study (see Method section), we are here interested in understanding if and how blind subjects can rely on their structure sense while solving an algebraic task, the accessibility of which is provided through digital tools. Our research question is: How can a blind subject rely on his structure sense while solving equations if supported by assistive technology?

METHODS

Due to the nature of our research question and because of the paucity of research literature on the topic, we have chosen to conduct an exploratory case study. Such design is recommended when the aim of the research is "to portray 'what it is like' to be in a particular situation, to catch the close up reality and 'thick description' […] of participants' lived experiences of […] a situation" (Cohen et al., 2007). Aiming at describing how a subject draws upon his/her structure sense, we interviewed an adult person who has a strong education in mathematics, to whom we refer with the pseudonym of Antonio. Antonio, who has a degree in Physics and has worked as a fellow researcher for 2 years, became blind four years ago due to a degenerative pathology. He learned to use LaTeX with speech synthesis as a visually impaired undergraduate student, 10 years ago.

We opted for a task-based interview: the interviewee was asked to select two equations among those proposed by Hoch and Dreyfus (2004) and to solve them. Because of the space limit, we will present only some excerpts from the solution process of one equation (Figure 1b), about which the solver was particularly talkative. The task was presented through a PDF file (Figure 1) which was implemented with the *Axessibility* package for LaTeX (Armano et al., 2018). By adding a single line of code to the source LaTeX file (line 2, Figure 1a), this package automatically inserts a hidden alternative text in the PDF document at each formula which is then accessible to screen readers (e.g., Jaws, NVDA). On Antonio's computer the screen reader NVDA was installed, allowing him to hear the read-aloud of LaTeX code. The LaTeX code for the equation in focus is shown in the box 'a' of Figure 1. In particular, the command \frac{}{} represents a fraction having as numerator the content of the first curly braces and as denominator the content of the following curly braces.

a

```
1 \documentclass{article}
2 \usepackage[accsupp]{axessibility}
3 \begin{document}
4  \[\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})\]
5 \end{document}
```

b

$$\frac{1}{4}-\frac{x}{x-1}-x=5+(\frac{1}{4}-\frac{x}{x-1})$$

Figure 1. Example of LaTeX code (box a) and corresponding compiled PDF (box b).

Aiming at a thick description (Bell & Kissling, 2019) we collected several sources of data including "speech acts; non-verbal communication; descriptions in low-inference vocabulary; […] recording of the time and timing of events; the observer's comments […]; detailed contextual data" as prescribed by Cohen et al. (2007, p. 405). This was realized by recording the interview including in the audio- and video-recording of the interviewee (through a webcam) and capturing the interviewee's computer screen. The second author of this report acted as interviewer and took personal notes during the interview. The video was transcribed verbatim by the first author integrating the transcription of 'what is said' with descriptions of 'what is done' (e.g., Table 1) – as recommended by Sfard (2008) – and with screenshots. Screenshots have been elaborated adding arrows representing the movements of the cursor (Figure 2); the final position of the cursor is represented by a vertical line. The three authors have analyzed this enriched transcript by coding each line with the six components of structure sense described in previous section. The coding process was discussed and reviewed among the three researchers until consensus was achieved.

RESULTS

The first excerpt refers to the first reading of the proposed equation. Antonio uses the functionalities of the Axessibility package (Armano et al., 2018) to hear the reading of the LaTeX code behind the PDF that we provided to him. Then, he decides to copy/paste the LaTeX code on the Microsoft Notepad application. The NVDA software reads the code out loud while his cursor navigates through the Notepad, as captured in the video.

Table 1. Enriched transcript of the first excerpt.

Line	What is said	What is done
1	Antonio: Let's see the structure. One fourth.	The cursor moves till the denominator of the first fraction and stops right before the closing curly brace.
2	Interviewer: While you are understanding the structure, would you like to tell it?	The cursor moves forward and stops after the minus sign.
3	Antonio: One fourth, yes.	The cursor moves back, before the minus sign.
4	Minus x over… Over x minus one. Minus x equals… Then there is the second member of the equation.	The cursor moves forward and stops right before the equal sign.
5	Five plus, open bracket. Then there is a bracket.	The cursor reaches the opening bracket.
6	One fourth inside the bracket.	The cursor moves back and forth over the \frac command.
7		The cursor reaches the end of the last fraction
8	Minus x over x minus one.	The cursor moves back and forth over the denominator of this fraction, then it goes back to the minus sign.
9	Closed bracket and that's it.	The cursor moves forward till the end of the equation.
10	Here I would start by working on the brackets.	

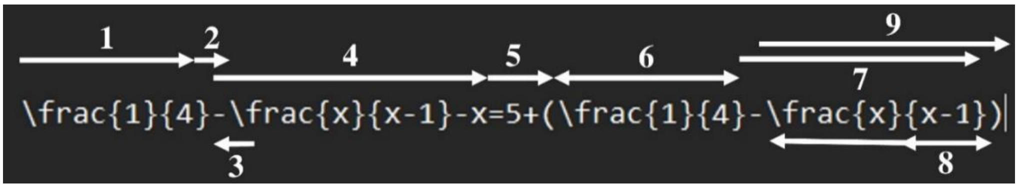

Figure 2. Cursor's movements during the first excerpt. Numbers are keyed to Table 1.

Despite not being familiar with the theoretical framework of this paper, Antonio starts by expressing the intent to understand 'the structure' (line 1). Since this strategy corresponds to reading the different sub-structures (the fractions, the parenthesis) of the equation, he is relying on the third component of structure sense, which is "divide an entity into sub-structures". Indeed, by analyzing the movements of his cursor, we can see that his attention is initially caught by the first fraction (lines 1-2). Then he analyzes the second fraction and stops when the left side of the equation finishes (line 4). The presence of the bracket is noticed (line 5) and then he goes back and forth over the two fractions within the brackets (lines 6-9). Then, we can notice that – even if the readers would normally force a left to right reading – Antonio analyses the structures of specific parts of the equation by realizing multiple readings of the identified sub-structures. This is particularly true for the fractions; their presence is highlighted by the LaTeX commands and the curly braces identifying the numerator and the denominator.

After this first excerpt, Antonio copies the whole equation on a second line in the Notepad. He recognizes that he can work inside the brackets first (component 5 of structure sense) and prepares the environment for doing so: he creates many blank spaces before the closing bracket (second line in Figure 3), then he states that he can calculated the least common multiple of the denominators (again component 5). He writes the denominator in the obtained blank space within braces (second line in Figure 3) and then adds a couple of braces before (third line) – so preparing the space for the numerator. He performs his calculations for the numerator between these braces (fourth and fifth line in Figure 3) and when he is done, he adds the command \frac before the braces (sixth line). Finally, he replaces the content of the brackets with the calculated fraction (last line in Figure 3). Then, he recognizes which manipulation he can realize and uses the braces as containers for organizing the structure of the result of such manipulations. After these manipulations, Antonio decides to work on the fractions on the first side of the equation, as shown in the excerpt in Table 2.

```
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1})
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1}        )
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1}        {4(x - 1)})
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1}         {}{4(x - 1)})
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1}         {x - 1 - 4x}{4(x - 1)})
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1}         {- 1 - 3x}{4(x - 1)})
\frac{1}{4}-\frac{x}{x - 1}-x=5+(\frac{1}{4}-\frac{x}{x - 1}         -\frac{3x + 1}{4(x - 1)})
\frac{1}{4}-\frac{x}{x - 1}-x=5+(-\frac{3x + 1}{4(x - 1)})
```

Figure 3. Different phases of Antonio's manipulations of the fractions in brackets.

Table 2. Enriched transcript of the second excerpt.

Line	What is said	What is done
11	Antonio: Ok. Now let's see what was here.	The cursor moves back to the beginning of the equation.
12	There was one fourth.	The cursor moves till the end of the second fraction
13	And then there was the same thing as in the brackets, but outside.	Then the cursor moves back to the beginning of the equation.
14	Thus… Thus, the result is the same of the other side because it's equivalent.	The cursor moves till the end of the left side of the equation.
15	Then I can copy this.	The cursor moves to the fraction on the right side of the equation, which is then selected.

16		The fraction is copied in a new following line on the Notepad.
17	Then I copy 'minus x equals'.	He types '-x='.
18	Well, I can copy and paste the second member. Yes, I copy and paste it.	He selects, copies, and pastes the right side of the equation.
19	Thus, since they are… the fractions are equal but opposite in sign… because if… then, I bring at the second member what is in the first member, they are equal and opposite. The result should be…	The cursor goes back to the beginning of the equation and then moves till the end of the right side.
20	x equals minus five.	He moves to a new line and write 'x=-5'.
21	Let me check…	The cursor goes back to the previous line and moves through the whole line, from left to right.
22	Yes, that should be the result.	

While reading the left side of the equation, Antonio recognizes the same structure of the expression within the brackets (line 13, component 2 of structure sense). Then, he understands that, instead of performing again all the manipulations shown in Figure 3, he can replace the fractions on the left side with the fraction calculated on the right side (lines 15-16, components 4 and 6), which was into the round brackets (Figure 1b). Having two identical sub-structures on the two sides of the equality, he decides to cancel them (lines 19-20, components 4 and 6). Hence, in this short excerpt we can notice many of the components of structure sense intervening and, considering the other parts of the transcript as well, we can observe all the six components of structure sense enacted.

DISCUSSION AND CONCLUSION

We can answer our research question by noticing that all the six components of structure sense have a role in Antonio's solving of the equation using the Microsoft Notepad and NVDA reader for manipulating the equation represented in LaTeX code. In particular, Antonio uses the reader to read (and re-read) *self-selected* portions of the equation instead of simply reading from left to right. The notepad is used to manipulate the equation both within the same line (differently than what we are used to do with paper and pencil, Figure 3) or connecting different lines (e.g., lines 13-18).

We have noticed that during the first reading (Table 1) the equation is divided into sub-structures (component 3 of structure sense) and then Antonio recognizes which manipulations are possible (component 5) on these sub-structures. As observed for seeing subjects, brackets play a relevant role in structuring the equation (Hoch & Dreyfus, 2004). However, in this specific case, we can notice that also the use of the LaTeX code – allowed by the Axessibility package (Armano et al., 2018) – may play an important role in structuring the equation into sub-structures, since the \frac command is often a place where the cursor stops. Furthermore, the LaTeX code becomes not only a tool for reading mathematics, but a tool for doing mathematics as well (in the sense of Alajarmeh et al., 2011). This is visible when the curly braces are used to organize the space of manipulation, distinguishing the numerator and the denominator of the algebraic fraction (Figure 3).

Antonio provides a telling example of how the LaTeX code can serve as a tool for symbolic manipulation with the interesting 'side effect' of being transformable in a PDF file which can be then read both by seeing students and visually impaired ones. As noted by Ahmetovic et al. (2021), LaTeX is a writing system used in all STEM disciplines, then its learning is both useful for academic achievement and for inclusivity. The study presented in this report suggests that the learning and use of LaTeX could promote and support structure sense especially for visually impaired students, but potentially not only.

We have also noticed that Antonio was able to recognize previously met (sub-) structures and use them to shortcut his manipulation (Table 1), so mobilizing many components (2-4-6) of structure sense. This recognition is realized after hearing the reading of the first part of the equation (line 13) by NVDA software; paraphrasing Radford's (2010) words, Antonio's ears have gone under a lengthy process of domestication through which they came to hear and recognize things according to an 'efficient' cultural mean. The reading of the equation and the memorized 'sound-track' acted as realizations of the equation in the sense of Sfard (2008) – being anything but visual. This fact corroborates that other senses than sight can successfully help not only in the rote manipulations of algebraic symbols, but in developing structure sense as well. This suggests that verbalization of the structure of algebraic expression may be an important step in the development of structure sense for blind people, but this is true for all the other students as well (Maffei & Mariotti, 2011). This observation strengthens what has been noted before in the case of LaTeX: adopting inclusive approaches to algebra teaching may be fruitful not only for impaired students, but for the whole class-group as well.

Surly, we must be cautious about the conclusion that we draw from a case study; in particular, we must consider that visual impairments are very different among them. For instance, Antonio was not completely blind during his high school studies and then he might rely on visual memories of algebraic expressions. Different results might be obtained in the case of students born blind and/or that are able to use Braille to read and write mathematical notations. Future developments of our research project will include subjects with different past histories about their disabilities and their learning of mathematics. Nevertheless, we hope that this report could offer a step forward in unveiling the (many) ways in which visual impaired solvers can successfully tackle algebraic equations.

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