# Teachers' awareness on combinatorics during Lesson Study discussion

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*This study focuses on the professional development of mathematics teachers collaborating within a Lesson Study. The research aim is to analyse how teachers' discussion can promote professional awareness. The data have been collected from a 6-cycle Lesson Study in an Italian secondary school, centred on the introduction to combinatorics. Mason's framework is the theoretical lens to qualitatively analyse teachers' discussion. This enabled the observation of how teachers' awareness of students' combinatorial thinking emerged and led to changes in teachers' practice. Collaborative reflection during the lesson study discussion phase appears to be critical for teacher professional development.*

*Keywords: Awareness, combinatorial thinking, lesson study, teacher professional development.*

## Introduction and Literature review

Collaboration among teachers is considered a key factor in improving teachers' education (Bakker et al., 2021) and teaching itself (Andriano & Manolino, 2023). In the frame of ICMI Study 25 (Borko & Potari, 2020), the academic community discussed how complex it can be to grasp and demonstrate the assertion. Lesson Study (LS) is one of the main sources for researchers to gain insight into teachers' attitudes towards working and learning, their mathematical knowledge for teaching, and teaching practices (Huang & Shimizu, 2016). In particular, a number of studies focus on analysing teachers' discourses as a window on teacher learning (e.g., Clivaz et al., 2023).

LS is a collaborative model for teachers' professional development inside the institutions (schools). LS is usually modelled as a cycle. Here, it is structured in the following phases: 1) Goal setting: formulate a "research" theme of common interest to teachers around a topic to be taught; 2) Planning: collaboratively prepare a Lesson Plan to implement the research lesson; 3) Observing: one group member teaches the lesson while other members observe; 4) Discussing: members discuss the data collected during the observation and revise the lesson. Then the cycle could be repeated multiple times. This structure has many variants and has been adapted in various countries with different features according to diverse cultural aspects (e.g., Seleznyov, 2018).

In Italy the first experiences were in primary school (Bartolini Bussi et al., 2017), nowadays LS is experienced at all school levels, from kindergarten to secondary school, and with prospective teachers (Capone et al., 2020; Manolino, 2021). Specifically, in the Italian experience, an essential emphasis is given to the cultural context in which the teaching-learning is situated. This is because LS was introduced as a teacher education proposal that, by leveraging cultural differences, stimulates teachers to reflect on themselves and their own habits (e.g., Arzarello et al., 2022).

LS is usually introduced by researchers to teachers, and emphasis is on the important role of the researcher as a cultural mediator (Ponte et al., 2018). In Italy, the researcher does not mediate cultural adaptation *a priori*, but uses it educationally as a stimulus for teachers' development (Ramploud et

al., 2022). In this study, there is an additional peculiarity: LS was introduced to a group of mathematics teachers by one of their colleagues, in the role of facilitator. The facilitator is author of the paper and collaborates with the researchers. The focus of this study is teachers' awareness, in the sense of Mason (2021), studied in the observation and discussion phases (3 and 4). Particularly we aim at analysing teachers' critical reflection during the discussion meetings, i.e., members' discussions after lessons observation. To conduct the analysis of a 6-cycles LS in an Italian secondary school focused on the introduction of combinatorics, in line with Andriano and Manolino (2023), we use the three-step theoretical lens based on the concept of awareness.

## Theoretical framework

The theoretical background is based on the concept of *awareness* as introduced by Gattegno (1987). Awareness triggers actions without our having to initiate them consciously, in the sense that they could lie under the surface of our consciousness. Using the term awareness in the sense of Gattegno, Mason (2021) proposed several valuable levels of awareness in educational contexts: 1) *Awareness-in-action*: the sensitivities to certain situations which enable action; 2) *Awareness-in-discipline*: awareness of awareness-in-action; 3) *Awareness-in-counsel*: awareness required in order to be sensitive to what others require to build their own awareness-in-action and in-discipline. As Mason sustains, being a teacher requires sensitivity to learners, and this is made possible because of awareness-in-discipline. According to Mason, awareness is educable and can be improved by the *discipline of noticing* "namely, imagining oneself acting appropriately in the future and using this to sharpen one's noticing in the moment, thus educating one's awareness." (Mason & Davis, 2013, p. 192). Conceptualization of noticing is varied: some authors define noticing solely as that to which teachers attend (Sherin et al., 2011), others consider noticing to involve two processes, attending, and making sense, and making sense for some researchers is solely interpreting, whereas for others involves interpreting and deciding how to respond (Jacobs et al., 2010). In this paper we focus on a single component of noticing, that of attending. To improve one's awareness, habits need to come to the surface, in the sense that, in education, teachers should observe their own habits from a distance, and use them as objects of their own reflection. This process can be enhanced when teachers work together and are part of a community where specific practices are shared (Wenger, 1988). Implementing Lesson Study, teachers are part of the same community and in preparing their Lesson Plan they have to share common practices and to define where to direct their attention depending on the research problem (Alston et al., 2011). During the observation of the lesson, teachers take distance from their habits, and pay attention to something new, they "notice" something. During the following discussion meeting, when teachers work together, these noticed things become objects of critical reflection. Critical reflection is a social process oriented to an action. "To notice is to make a distinction […] that distinction enables an appropriate action to be initiated. […] The coupling of a distinction and an action is what Gattegno refers to as an awareness." (Mason, 2021, p. 240). Based on the previous considerations, Andriano and Manolino (2023) proposed to frame the analysis of awareness in teachers' discussions and reflections in a qualitative study divided into three steps: *noticing, objectifying,* and *acting*. The terms are used with the following meaning:

a) *Noticing*: teachers notice something, there is a *shift of attention*. It can take place below the level of consciousness; it is like stepping into a new point of view.

b) *Objectifying*: the noticing becomes an object of discussion. Teachers take distance from their habits and their actions become objects of collaborative reflection. It is a social process, the moment of *critical reflection*.

c) *Acting*: "The important question is whether discerning that distinction enables an appropriate action to be initiated" (Mason, 2021, p. 240). Critical reflection is oriented to action. It is a transformative process, i.e., a collaborative moment of teachers' development, where teachers agree on acting and change their own practice.

According to these theoretical elements, our research question is: How do teachers improve their awareness when involved in a professional practice inspired by Lesson Study?

## Context and Method

This study takes place in an Italian secondary scientific high school (grades 9-13) and it is part of a larger, three-year research project aimed at introducing LS in a group of mathematics teachers and studying the teachers' processes. These teachers volunteered to be introduced to LS by a colleague working in the same school (the first author of this paper), who acted as facilitator in disseminating LS main ideas and in organising the different phases of a cycle. Since the facilitator had the opportunity to participate as an observer in some LS research lessons and discussions in Shanghai, during a period spent in China, coming back to Italy she first shared her experience with the Mathematics Education researchers of the Department of Mathematics "G. Peano" of the University of Torino and Bari (the other authors), then introduced LS to her colleagues. A decision arising from the meetings with the researchers was to introduce LS to colleagues as an experience "in action".

The first year of the project was devoted to finding a suitable structure for implementing LS in the Italian institutional context, in line with the Italian background of Mathematics education (Arzarello & Bartolini Bussi, 1998), and with the Cultural Transposition framework (Ramploud et al., 2022). During the 2019/20 school year, only one LS cycle was implemented, due to the COVID-19 pandemic. In 2020/21, multiple cycles of LS were implemented. The data for this paper come from these cycles. The LS group consisted of 5 teachers with 10-20 years of teaching experience in secondary school. One of the five teachers, the first author, served as the LS facilitator. The group had taken part in one LS cycle in the previous year. Three teachers taught in grade 12, one teacher taught in grade 10 and one teacher taught both grade 10 and 12. The experiment involved four grade-12 classes and two grade-10 classes. The LS followed this structure:

- 1 online "Goal Setting" meeting to choose the research theme and the topic (henceforth referred to as M1);
- 2 "Planning" meetings to design the research lesson and write the Lesson Plan (M2, M3).
- 6 research lessons: 4 lessons in grade 12 (L1, …, L4) taught by teacher $T_1$, …, $T_4$; 2 lessons in grade 10 (L5, L6) both taught by teacher $T_5$ (there were 6 lessons because all teachers wished to teach the lesson and they were allowed to).
- 6 "Discussing" meetings, each following the respective research lesson in the same day (from here on, discussion meetings are referred to as DM1, …, DM6).
- 1 online meeting to summarise the project conclusion (M4).

Due to the pandemic, the classes involved in the project were only allowed to have 50% of students physically present in the classroom. The students were split into two groups: one attending class physically, and the other attending online. Therefore, in M1, teachers choose two research problems: 1) analyse students' combinatorial thinking; 2) compare the interactions among the students and between students and teachers according to the different modes of attendance: physical and online. In this research we focus on the first research problem. In M1, it was decided to plan one lesson for

the introduction to combinatorics because it is a suitable topic both for grade 10 and 12 and furthermore the didactics of combinatorics is as challenging for students as it is for teachers (Batanero & Borovcnik, 2016). In M2 and M3, the teachers designed the research lesson and wrote the Lesson Plan. The focus of the lesson was a group activity on a problem followed by whole class discussion. The lesson included three phases:

1) Introduction: the teacher introduces the topic starting from the concept of permutation.
2) Working group: students work in groups on a problem.
3) Institutionalisation: a whole class discussion.

The problem for the group activity was chosen in M2 and M3. Teachers were looking for a problem with a simple formulation for straightforward understanding, and at the same time engaging and challenging for all students. The teachers agreed upon the Alpinists' Problem reported on the textbook:

> 8 alpinists need to rope up in line to walk across a glacier. Two of them are beginners and cannot be placed neither in the first nor in the last position. How many ways are there to form the line?

During the lesson, students were given the problem and tasked to work in groups, without teacher's help. In-person students worked in groups of three. Online students worked in bigger groups in Zoom breakout rooms. Two teachers observed the physical groups, and two teachers observed the students in the Zoom breakout rooms. All LS meetings and lessons were video recorded. Teachers' observation notes, as well as student samples and interactive boards, were collected and saved during the lessons.

**Method**

The aim of this study is to analyse teachers' critical reflection during collaborative moments of discussion. The theoretical framework justifies the use of qualitative analysis. Therefore, the videos, the transcripts of teachers' words and teachers' observation notes are the basis of our data. To carry out the analysis we viewed many times the videos and we analysed transcripts of meeting discussions using the developed theoretical lens (noticing, objectifying, acting). First of all, we identified in the videos some moments of noticing, i.e., some moments in the discussion when a teacher reported to have noticed something, a shift of attention. Then we kept track of these noticings in the same and in the following meetings, considering these questions: do they evolve in a critical discussion, or are they left forever, or are they left and then return in following meetings? Is there a moment of critical reflection? Does this critical reflection result in teachers' action? In this way we collected some episodes where we can distinguish the three steps: noticing, objectifying, and acting. In this paper we focus on the analysis of one episode of noticing that evolves in a critical reflection and results in teacher action.

## Results and data analysis

In this section, we analyse, through the developed theoretical lens (noticing, objectifying, acting), one episode where students' combinatorial thinking becomes objects of inquiry for teachers. The following transcripts are taken from the third "Discussing" meeting (DM3) that took place one hour after lesson 3 (L3) delivered by teacher T3. Teacher T3 introduced the meeting with some considerations about the lesson. Then every teacher added some comments.

**Noticing**

T5: At a certain point, in the problem but also in the final solution, it was said that you can move the letter to a different position, or you can fix the position and choose a letter to put in. Because it is different… (DM3, 11:45)

T5 notices something, makes a distinction, and that distinction is expressed in words. T5 notices a different way to see the permutations. The written calculation is the same, but it is seen in a different way. T5 notices that the same occurs in the whole class discussion. Even though the calculation is the same, 6x5x6!, students' reasoning in solving the problem was different from the one that the teacher used to summarise. The object of noticing becomes the object of critical reflection. Teachers start discussing combinatorial reasoning.

**Objectifying**

During the discussion, teachers became aware of different ways of solving the problem. Moreover, they became aware that summarising students' strategies could change the original idea. The discussion goes on comparing different solutions of the alpinists' problem proposed by the students.

T2: Yes, you can see it in two ways.
T5: Actually, also in the final solution. You had summed up the reasoning in a different way compared to what the student said. Writing 6x5, the student didn't say: 'for the first and the last place I have 6 and 5 possibilities'. He said: 'beginners can stay in 6 places. I take the first, he can stay in one of the six central places.
T1: Yes, right!
T5: Then one takes the second beginner and can put him in 5 places. So, he [the student] took the places and put in them the alpinists.
T3: Right, true!
T5: The student said this, then you summed up in a different way that was the way we solved the problem [when preparing the LP]. … (DM3, 12:10)
T3: Often, when a person has an idea, s/he doesn't listen carefully to the solution, and doesn't have time to understand deeply. I didn't get this solution, you [T5] are telling me now. I saw the right calculation; the student was explaining, and I summed up to the class in the way I understood it. This thing is interesting. (DM3, 13:14)

Combinatorial thinking becomes the object of critical reflection. During the Lesson Plan design, teachers' solution of the problem followed this reasoning: you have 8 places for the 8 alpinists. In the first place you can choose among 6 experts, in the last place you can choose among 5 experts, and then you must fill the 6 central places with the remaining 6 alpinists. In formula: $D_{6,2} \cdot P_6 = 6 \cdot 5 \cdot 6!$. During the lessons, many other strategies were found by the students. For example, in the transcript, T5 observed that the solution proposed by a group had the same calculation but a different reasoning: in the first solution (the teachers' one) the objects are the alpinists, in the second the objects are the positions in the rope. The noticing of T5 becomes an object of discussion. During the moment of critical reflection, T3, who delivered the lesson, became aware of the different strategies proposed by the student during the lesson and of the fact that his way of summarising had modified students' reasoning: "I didn't get this solution, you are telling me now. I saw the right calculation, the student was explaining, and I summed up for the class in the way I understood it". Other groups of students proposed strategies by subtraction, i.e., $8! - 2 \cdot 7! - 2 \cdot 7! + 2 \cdot 6!$ or $8! - 26 \cdot 6!$. These, and other solving strategies observed by teachers in the group work, were discussed.

T4: I really appreciate your explanation about the solution by subtraction, 8! minus…. I would not have been able to do it. And probably, for my idea, I wouldn't have insisted on this kind of solution, I would have insisted on the other way. I would have dropped the solution by subtraction, pretending not to see it. And probably this is wrong, because we don't all have the same ways of reasoning. It is not respectful of students' ways of working. (DM3, 18:01)

During the discussion T4 reports the fact that, observing the lesson delivered by another teacher, realised that in the same situation they would have acted differently: "I wouldn't have insisted on this kind of solution. I would have insisted on the other way. I would have dropped the solution by subtraction, pretending not to see it". The critical reflection made it possible for T4 to envision other ways of acting: "probably this is wrong, because we don't all have the same ways of reasoning".

**Acting**

The critical reflection is oriented to an action. The critical reflection made it possible for T4 to have other action coming to mind in a similar situation. Two days after this discussion, the next lesson (L4), was delivered by T4. In the group activity, 3 groups found the right solution with different strategies. Two of them by subtraction. T4 didn't "drop" it. On the contrary, during the whole class discussion, T4 asked students to explain their strategies.

T5: It was really interesting because there was more than one method by subtraction. … One group, instead of taking 8! and then subtracting all the non-acceptable possibilities, and in this way, you have first to subtract then to add, and it is difficult. They have considered the position, not the place. They said: 'I must subtract: when the first and the last are beginners, so 6!; then when the last and the first [are beginners]; then when the first is a beginner, and the other is in one of the central positions: 6 times 6!.' … In this way they have only subtracted 2·13·6!. …

T1: Yes! 12 and 12 and 2, 26·6!, (All teachers nodding)

T4: Yes! And they were also able to explain. Their idea was clear. Then a student said that she didn't understand, and I asked the group to explain, and they explained clearly. (DM4, 20:04)

Conducting the lesson, T4 showed a different awareness of combinatorial thinking. Two days ahead, during the discussion, T4 said "I would have dropped the solution by subtraction, pretending not to see it". On the contrary, conducting the lesson T4 had recognized combinatorial reasoning by subtraction and had invited the group to explain the reasoning to the whole class. The new level of awareness is recognizable in two ways. On the one hand, the teacher was able to recognize different ways of solving the problem, different combinatorial thinking, and therefore was able to lead the whole class discussion asking students to explain the different solutions. On the other hand, the teacher was aware of the fact that summarising students' reasoning could lead to misinterpretations, therefore T4 asked students to explain their own reasoning. Moreover, the discussion among teachers during this discussion meeting (DM4) shows a different level of awareness of combinatorial thinking of the whole group of teachers. They all took part in the discussion, showing that they were able to recognize the different combinatorial thinking.

## Discussion

The focus of this paper was to study how teachers' discussion in Lesson Study can promote changes in teachers' awareness and in their practice. To analyse collaborative moments of critical reflection in teachers' meetings, and to capture how awareness can promote changes in teachers' practice, we proposed a theoretical lens structured in three steps:

a) *Noticing*: teachers notice something, they make a distinction and express it in words.
b) *Objectifying*: the object of noticing becomes an object of critical reflection.
c) *Acting*: critical reflection is oriented to action. There is a change in teachers' practice.

This framework allowed us to examine and interpret teachers' discussion. During the LS, teachers improve their awareness on combinatorial thinking. They become aware of the wide variety of strategies that students use in solving the problem, and this enables them to effectively lead the whole

class discussion on the problem. The improvement in teachers' awareness regards two different aspects. The first one concerns combinatorial thinking: teachers become able to recognize different strategies and argumentations supporting the same written calculation. This awareness lets teachers choose the appropriate task, direct students' attention and effectively lead the whole class discussion enhancing students' thinking and contributions. The second improvement in teachers' awareness regards the communicative strategies used during a whole class discussion. One of them is summarising students' strategies and during the discussion they became aware that summarising students' strategies could change the original idea. These improvements in awareness have been reached through teachers noticing and moments of collaborative critical reflection, and it results in actions or changes in teachers' practice. This was possible because of the participation of teachers both in the observation of the lessons and in group discussions, which is typical of LS. In this paper only one episode is analysed. We need to expand our research analysing further episodes and identifying different aspects of teachers' awareness, especially focusing on the moments when noticing leads to objectifying and acting, and when it does not. Moreover, further research is needed in order to analyse which are the characteristics of the applied design of the LS which emphasise this learning of teachers.

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