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On a Linear Differential Game of Pursuit with Integral Constraints in ℓ^2

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Abstract: In this paper, we study the stability, controllability, and differential game of pursuit for an infinite system of linear ODEs in ℓ^2. The system we consider has a special right-hand side, which is not diagonal and serves as a toy model for controllable system of infinitely many interacting points. We impose integral constraints on the control parameters. We obtain criteria for stability and null controllability of the system. Further, we construct a strategy for the pursuer that guarantees completion of the pursuit problem for the differential game. To prove controllability we use the so called Gramian operators.

Keywords: differential equations in Hilbert spaces; mild solutions; null control problem; optimal control; pursuit problem; linear operators

MSC: 49N05; 93C15

1. Introduction

Control problems in Banach or Hilbert spaces arise naturally when modelling processes such as heat transfer and fluid dynamics. By now it is well-established subject many works have been devoted to it (see, for example, the monographs [1–6]). Although the general theory is well developed, the area is still a very active field of research with a diverse range of problems (see, for example [7–9] and references therein for controllability and observability of systems in Banach/Hilbert spaces).

Control problems for partial differential equations sometimes maybe reduced to control problems for infinite systems of ODEs. For example, in the works [10–12], it is shown that in certain cases, the controllability of the system described by an evolutionary equation may be obtained in terms of finite-dimensional approximations of the system, and the authors show the approximate solutions converge to a solution of the initial control problem. In the above works, the control problem for PDEs reduces to an infinite system of linear ODEs with a very special form: a diagonal right-hand side, making it easy to prove the convergence of the approximations. On the other hand, the type of infinite system obtained from a PDEs depends on boundary conditions besides the system itself and, therefore, does not always reduce to an ODE with a diagonal right-hand side. Here, in this work, we put step forward in this direction and consider an infinite system of ODEs whose right-hand side is not of a diagonal form. However, the situation in infinite dimensional systems differ drastically from their finite dimensional counterparts, since in a very simple set up, like ours here, the spectrum of the linear operator defined by the right-hand side might become very large and contain a disk. This makes the problem really

challenging. Moreover, many processes in applications are well suited for modelling as a system of infinite number of ODEs. Therefore, infinite systems attracted enormous amount of attention. See, for example, the monographs [3,4,6] and references therein.

For example, in [13] the existence of approximation schemes for certain linear systems with a quadratic cost is studied. These schemes converge, but the approximating controls do not stabilize the original system, and the costs do not converge. A different example was considered in [14], where the stability and controllability of a system in the form of an infinite Jordan block, with $\lambda \in \mathbb{R}$ on the main diagonal is considered.

Differential games in this setting have also been studied, with works such as [15–20] considering pursuit-evasion games for systems in infinite-dimensional phase spaces. In certain cases, optimal strategies for players were constructed within appropriate strategy classes. However, the main restriction on the system, i.e., having a diagonal operator on the right-hand side, remains unchanged.

In this paper, we relax this condition and consider a controllable system of linear ODEs in Hilbert space ℓ^2. Our long-term goal is to study a system of countably many weakly interacting points and this work is a step in that direction. As a toy model, we restrict ourselves to the case when all the points have same dynamical characteristics, and the motion of each particle is simple and depends on the positions of finitely many other particles. Thus, the right-hand side of the has a special form: the main diagonal contains $\lambda \in \mathbb{R}$ and finitely many non-zero elements, whose sum is equal 1. Consequently, any finite-dimensional representation of the system is asymptotically stable if $\lambda < 0$. On the other hand, the infinite system is only stable when $\lambda \leq -1$, and if $\lambda > -1$, certain solutions will increase exponentially. This indicates that there are distinctions between finite and infinite systems. Another main characteristic of this work is that we provide an explicit form of control functions that can stabilize the system through the use of Gramian operators.

The rest of the paper is organised as follows. In Section 2, we describe the problem and state the main results. In Section 3, we prove the main results. Namely, in Section 3.1, we show the global asymptotic stability. In Section 3.2, we prove the global null controllability. Section 3.3 of Section 3 is devoted to the study of the differential game problem of pursuit. Finally, the results and further generalisations are discussed in Section 4.

2. Main Results

Recall that

$$\ell^2 = \{x = (x_1, x_2, \ldots) \mid x_k \in \mathbb{R}, \ \sum_{k=1}^{\infty} x_k^2 < \infty\}$$

is a Hilbert space with its natural inner product and norm

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k, \quad \|x\|_2^2 = \sum_{k=1}^{\infty} x_k^2, \quad x, y \in \ell^2.$$

We consider the following coupled infinite system of ODEs:

$$\dot{x}_k = \lambda x_k + a_1 x_{k+1} + a_2 x_{k+2} + \ldots + a_N x_{k+N}, \ x_k(0) = x_{k,0}, \tag{1}$$

where $\lambda \in \mathbb{R}$, $N \in \mathbb{N}$ are constants, $\sum_{i=1}^{N} a_i = 1$, $a_i \in \mathbb{R}_+$ and $x_0 = \{x_{k,0}\}_{k \in \mathbb{N}} \in \ell^2$.

The right-hand side of (1) defines a linear operator $A : \ell^2 \to \ell^2$:

$$Ax = \{\lambda x_k + a_1 x_{k+1} + a_2 x_{k+2} + \ldots + a_N x_{k+N}\}_{k \in \mathbb{N}}$$

and hence, adopting the notation $x = \{x_k\}_{k \in \mathbb{N}}$ we can rewrite (1) in the form

$$\dot{x} = Ax, \ x(0) = x_0. \tag{2}$$

Further, letting Id to denote the identity map and $E : \ell^2 \to \ell^2$ the shift map, i.e., $[Ex]_i = x_{i+1}$ for $x = (x_1, x_2, \dots) \in \ell^2$, we write

$$A = \lambda \, \mathrm{Id} + \sum_{i=1}^{N} a_i E^i.$$

It is not difficult to see that $\|E^i\| = 1$ for all $i \in \mathbb{N}$ and thus A is a bounded linear operator. Indeed,

$$\|A\| = |\lambda| + \sum_{i=1}^{N} a_i \cdot \|E^i\| \leq |\lambda| + 1.$$

The above bound also implies that the operator

$$e^{tA} := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!},$$

is well a defined and bounded in ℓ^2 for every $t \in \mathbb{R}$. Further, it is easy to check that e^{tA} forms a continuous group. Thus, the solution of (2) can be written in the form

$$x(t) = e^{tA} x_0.$$

Also, the solution of the corresponding inhomogeneous equation

$$\dot{x} = Ax + u, \ x(0) = x_0, \tag{3}$$

is given by

$$x(t) = e^{tA} x_0 + e^{tA} \int_0^t e^{-sA} u(s) ds,$$

where u is a control parameter such that $u \in L^2([0, T], \ell^2)$ for any $T > 0$.

The first result is about the necessary and sufficient condition for the system (1) to be asymptotically stable.

Definition 1. *System (1) is called globally asymptotically stable if*

$$\lim_{t \to \infty} x(t) = 0$$

for the solution $x(t)$ of (1) with any initial condition $x_0 \in \ell^2$.

Recall that asymptotic stability plays a crucial role in null-controllability. Therefore, the following theorem is an important step in the direction of mull-controllability.

Theorem 1. *Let $x(t)$ be the solution of (1) with an initial condition $x_0 \in \ell^2$. For every $x_0 \in \ell^2$ and for every $t \in \mathbb{R}$ holds $\|e^{tA} x_0\|_2 \leq e^{(1+\lambda)t} \|x_0\|_2$. Moreover, the system (1) is asymptotically stable if and only if $\lambda \leq -1$.*

In fact, a more general statement is true.

Remark 1. *It is natural to consider the case $\sum_{i=1}^{N} a_i = M$. In this case, our proofs show that system (1) is asymptotically stable if $\lambda < -M$, but when $\lambda \geq -M$ the origin is an unstable fixed point of (1). Since the changes are minor, we do not pursue this generalisation here. We discuss further generalisations in Section 4.*

Let $\rho > 0$ be a positive number and let $T > 0$ be a sufficiently large fixed number. A function $u : \mathbb{R} \to \ell^2$ is called admissible on $[0, T]$ if

$$\|u\|^2 = \int_0^T \|u(t)\|^2 dt \leq \rho^2. \tag{4}$$

Definition 2. *We say that system* (3) *is null controllable in time T from $x_0 \in \ell^2$ if there exists an admissible control $u : \mathbb{R} \to \ell^2$ such that the solution of $x(t)$ of* (3) *satisfies $x(T) = 0$.*

System (3) *is called locally null controllable if there exists $\delta = \delta(\rho) > 0$ such that* (3) *is null controllable from $x_0 \in \ell^2$ with $\|x_0\| \leq \delta$.*

System (3) *is called globally null controllable or null controllable in large provided it is null controllable from any $x_0 \in \ell^2$.*

The next result is about the null controllability of the system (3).

Theorem 2. *For any $\lambda \in \mathbb{R}$, system* (3) *is locally null controllable. If $\lambda \leq -1$, System* (3) *is globally null controllable. Further if $\lambda < -1$, then starting the transfer to zero is happens within time $\tau < \|x_0\|_2^4 / \kappa \rho^4$ from any initial state $x_0 \in \ell^2$, where κ is a constant that does not depend on x_0.*

We also study the corresponding differential game:

$$\dot{x} = Ax + u - v, \ x(0) = x_0, \tag{5}$$

where $u : \mathbb{R} \to \ell^2$ is a control parameter of the pursuer, $v : \mathbb{R} \to \ell^2$ is a control parameter of the evader. We assume that both u and v are integrable functions such that

$$\|u\|^2 = \int_0^T \|u(t)\|_2^2 dt \leq \rho^2, \tag{6}$$

$$\|v\|^2 = \int_0^T \|v(t)\|_2^2 dt \leq \sigma^2. \tag{7}$$

for some $\rho > 0$, $\sigma > 0$ and sufficiently large $T > 0$. Control functions satisfying the above constraints are called admissible controls of the pursuer and evader on $[0, T]$, respectively.

Thus, the initial value problem (5) has a unique solution given by

$$x(t) = e^{tA} x_0 + e^{tA} \int_0^t e^{-sA} (u(s) - v(s)) ds.$$

We prove that the pursuit can be completed.

Theorem 3. *If $\rho > \sigma$, then there exist $T > 0$ and a strategy $u(\cdot)$ of the pursuer, such that $x(\tau) = 0$ holds at some $\tau \in [0, T]$ for any admissible control of the evader. Hence, the differential game* (5) *can be completed within the time T.*

3. Proofs

In this section, we prove the above theorems. First, we show the asymptotic stability, then we prove theorem about controllability, and finally we prove construct a strategy for the pursuer that gives guaranteed time to finish the pursuit.

3.1. Asymptotic Stability

Here, we give the proof of Theorem 1.

Proof of Theorem 1. By utilising the properties of the norm, we obtain

$$\|x(t)\|_2 \le \|e^{tA}\| \cdot \|x_0\|_2 = e^{\lambda t} \|e^{t \sum_{i=1}^{N} a_i E^i}\| \cdot \|x_0\|_2$$

$$\le e^{\lambda t} e^{t\| \sum_{i=1}^{N} a_i E^i\|} \cdot \|x_0\|_2 \le e^{\lambda t} e^{t \sum_{i=1}^{N} |a_i| \|E^i\|} \cdot \|x_0\|_2$$

$$\le e^{\lambda t} e^{t \sum_{i=1}^{N} |a_i|} \cdot \|x_0\|_2 = e^{(\lambda+1)t} \cdot \|x_0\|_2.$$

If $\lambda < -1$, then according to the last inequality we have $\lim x(t) = 0$ as $t \to \infty$. The above argument does provide useful information if $\lambda = -1$. Therefore, we need to use another argument to show that the system (1) is asymptotically stable in this case.

Our argument is based on the fact that e is weakly contracting operator in ℓ^2, i.e., $\|E^j x_0\| \to 0$ as $j \to \infty$. We have following equality for the solution $x(t)$, which started from $x_0 = (x_{10}, x_{20}, \ldots) \in \ell^2$

$$\|x(t)\|_2 = e^{-t} \|e^{t \sum_{i=1}^{N} a_i E^i} x_0\| = e^{-t} \| \sum_{j \ge 0} \frac{t^j}{j!} (\sum_{i=1}^{N} a_i E^i)^j x_0\|. \tag{8}$$

Since $\|E^j\| = 1$ for all $j \in \mathbb{N}_0$, we obtain

$$\|(\sum_{i=1}^{N} a_i E^i)^j x_0\|_2 \le \|(\sum_{i=1}^{N} a_i E^{i-1})^j\| \cdot \|E^j x_0\|_2$$

$$\le \left(\sum_{i=1}^{N} |a_i| \cdot \|E^{i-1}\|\right)^j \cdot \|E^j x_0\|_2 = \left(\sum_{i=1}^{N} |a_i|\right)^j \|E^j x_0\|_2 = \|E^j x_0\|_2.$$

Recalling the fact $\|E^j x_0\| \to 0$ as $j \to \infty$, from the above inequality we obtain $\|(\sum_{i=1}^{N} a_i E^i)^j x_0\|_2 \to 0$ as $j \to \infty$. Consequently, for any $\varepsilon > 0$ there exists $M \in \mathbb{N}_0$ such that $\|(\sum_{i=1}^{N} a_i E^i)^j x_0\|_2 < \frac{\varepsilon}{2}$ for all $j \ge M$. By fixing such an M and applying

$$\|(\sum_{i=1}^{N} a_i E^i)^j x_0\|_2 \le \|E^j x_0\|_2 = \|x_0\|_2 \quad j \in \mathbb{N}_0,$$

from (8), we obtain

$$\|x(t)\|_2 \le e^{-t} \sum_{j=0}^{M} \frac{t^j}{j!} \|(\sum_{i=1}^{N} a_i E^i)^j x_0\|_2 + e^{-t} \sum_{j=M+1}^{\infty} \frac{t^j}{j!} \|(\sum_{i=1}^{N} a_i E^i)^j x_0\|_2$$

$$\le \|x_0\|_2 e^{-t} \sum_{j=0}^{M} \frac{t^j}{j!} + \frac{\epsilon}{2} e^{-t} \sum_{j=0}^{\infty} \frac{t^j}{j!} \le \|x_0\|_2 C_M e^{-\frac{t}{2}} + \frac{\epsilon}{2}. \tag{9}$$

It is easy to see that the choice of M and hence C_M does not dependent on t. Therefore, (9) implies that there exists $t(\epsilon) > 0$ such that $\|x_0\|_2 C_M e^{-\frac{t}{2}} \le \frac{\epsilon}{2}$ for all $t \ge t(\epsilon)$. Hence, $\|x(t)\|_2 \le \epsilon$ for all $t \ge t(\epsilon)$, i.e., (1) is asymptotic stable when $\lambda = -1$.

Assume that $-1 < \lambda < 0$. This is sufficient, since system (1) is not stable for $\lambda > 0$. This is readily seen by the following example.

Example 1. *Let $\theta = -\lambda + \frac{|1+\lambda|}{2} \in (0,1)$ and define $\Theta = (1, \theta, \theta^2, \theta^3, \cdots)$. Obviously, $\Theta \in \ell^2$ and $E^i \Theta = \theta^i \Theta$. Consequently, $e^{tE^n}\Theta = e^{t(a_1\theta + \dots + a_N\theta^N)}\Theta$. Since $\theta > 0$ and $a_i \geq 0$ for all $i = 1, \dots, N$. we have*

$$\|e^{tA}\Theta\|_2 = e^{t\theta}\|\Theta\|_2 \to +\infty.$$

Consequently, the system (1) is not stable for $\lambda > -1$. This completes the proof. $\square$

3.2. Null Controllability

In this section, we show that System (3) is null controllable. The following simple lemma from operator theory will be useful.

Lemma 1. *Let $L : \mathcal{H} \to \mathcal{H}$ be a self-adjoint operator defined on a Hilbert space $(\mathcal{H}, \|\cdot\|)$. Assume that there exists $\kappa > 0$ such that $\|Lx\| \geq \kappa\|x\|$ for all $x \in L$. Then, L is invertible and $\|L^{-1}\| \leq \kappa^{-1}$.*

The following lemma is the main technical tool in the proof of Theorem 2.

Lemma 2. *The Gramian operator is defined by*

$$W(\tau) = \int_0^\tau e^{-sA} \cdot e^{-sA^*} ds,$$

is bounded, self-adjoint, and positive definite for any $\tau \in \mathbb{R}$. Furthermore, for every $x \in \ell^2$ there exists $\kappa > 0$ such that $\|W(\tau)x\|_2 \geq \kappa\|x\|_2$.

Proof. Using the properties of adjoint operator and taking into account E^js mutually commute, we have

$$e^{tA^*} = e^{-t(\lambda \operatorname{Id}^* + a_1(E^1)^* + a_2(E^2)^* + \dots + a_N(E^N)^*)} = e^{-t(\lambda \operatorname{Id}^* + a_1(E^*)^1 + a_2(E^*)^2 + \dots + a_N(E^*)^N)}.$$

Further, we have

$$e^{tA}e^{tA^*} = e^{-t(\lambda \operatorname{Id} + a_1E^1 + a_2E^2 + \dots + a_NE^N)} \cdot e^{-t(\lambda \operatorname{Id}^* + a_1(E^*)^1 + a_2(E^*)^2 + \dots + a_N(E^*)^N)}$$

$$= e^{-2t\lambda} \cdot e^{-ta_1(E^1 + (E^*)^1)} \cdot e^{-ta_2(E^2 + (E^*)^2)} \cdot \dots \cdot e^{-ta_N(E^N + (E^*)^N)}.$$

Furthermore,

$$|\langle W(\tau)x, \mathbf{y}\rangle| \leq \left\| \int_0^\tau e^{-2t\lambda} e^{-t\sum_{n-1}^N a_n(E^n + (E^*)^n)} x dt \right\|_2 \cdot \|\mathbf{y}\|_2 \leq$$

$$\leq \int_0^\tau e^{2t(1-\lambda)} dt \cdot \|x\|_2 \cdot \|\mathbf{y}\|_2 \leq M(\tau) \cdot \|x\|_2 \cdot \|\mathbf{y}\|_2, \tag{10}$$

where the $M(\tau)$ is a constant and depends only on τ.

Denote by $e_{ij}(t)$, $i, j \in \mathbb{N}$ elements of of the matrix $e^{t\sum_{n=1}^N a_n(E^n + (E^*)^n)}$, then for $x, y \in \ell^2$, we can write

$$\langle W(\tau)x, y\rangle = \sum_{i=1}^\infty \sum_{j=1}^\infty \int_0^\tau e^{-2t\lambda} e_{ij}(-t)x_jy_i dt. \tag{11}$$

Equation (10) implies that the right-hand side of (11) is absolutely convergent. Thus, we can change the order of summation in it:

$$\langle W(\tau)x, y\rangle = \langle x, W(\tau)y\rangle.$$

Hence, $W(\tau)$ is self-adjoint for every $\tau \in \mathbb{R}$. Further, since the series

$$\sum_{i=1}^{\infty} e_{ij}(-t)x_j y_i \quad \text{and} \quad \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} e_{ij}(-t)x_j y_i$$

are uniformly convergent in the interval $[0, \tau]$, we can change the order of summations and integration:

$$\langle W(\tau)x, x \rangle = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \int_0^{\tau} e^{-2t\lambda} e_{ij}(-t)x_j x_i dt$$

$$= \int_0^{\tau} \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} e^{-2t\lambda} e_{ij}(-t)x_j x_i dt = \int_0^{\tau} \langle e^{t\sum_{n=1}^{N} a_n(E^n + (E^*)^n)}x, x \rangle dt \qquad (12)$$

$$= \int_0^{\tau} e^{-2t\lambda} \langle e^{-t\sum_{n=1}^{N}\left(a_n(E^*)^n\right)}x, e^{-t\sum_{n=1}^{N}\left(a_n(E^*)^n\right)}x \rangle dt.$$

This implies $\langle W(\tau)x, x \rangle > 0$ for any $x \neq 0$ i.e., $W(\tau)$ is positive definite. Further, in the last line of (12) the function under the integral is positive, for every $\varepsilon \in [0, \tau]$ we have

$$\langle W(\tau)x, x \rangle \geq \int_0^{\varepsilon} e^{-2t\lambda} \langle e^{-t\sum_{n-1}^{N}\left(a_n(E^*)^n\right)}x, e^{-tE^*}e^{-t\sum_{n-1}^{N}\left(a_n(E^*)^n\right)}x \rangle dt.$$

By the definition for the operator $e^{-t\sum_{n=1}^{N} a_n E^n} \cdot e^{-t\sum_{n=1}^{N} a_n(E^*)^n}$ and for sufficiently small $\varepsilon > 0$ and $t \in (0, \varepsilon)$, we have

$$e^{-t\sum_{n=1}^{N} a_n E^n} \cdot e^{-t\sum_{n=1}^{N} a_n(E^*)^n} = \text{Id} - t\left(\sum_{n=1}^{N} a_n E^n + \sum_{n=1}^{N} a_n(E^*)^n\right) + \delta(t), \qquad (13)$$

where $\delta(t)$ is and operator whose ℓ^2 norm converges to 0 when $t \to 0$. Thus, choosing $\varepsilon > 0$ sufficiently small such that $\|\delta(t)\| < \varepsilon$ we have

$$\left\langle \text{Id} - t\left(\sum_{n=1}^{N} a_n E^n + \sum_{n=1}^{N} a_n(E^*)^n\right) + o(t)x, x \right\rangle > |1 - \varepsilon - t| \cdot \|x\|_2^2.$$

From the latter equation and (13), we have

$$\int_0^{\varepsilon} e^{-2t\lambda} \langle e^{-t\sum_{n=1}^{N} a_n E^n} \cdot e^{-t\sum_{n=1}^{N}\left(a_n(E^*)^n\right)}x, x \rangle dt$$

$$= \int_0^{\varepsilon} e^{-2t\lambda}|1 - \varepsilon - t|\|x\|_2^2 dt \qquad (14)$$

$$> (1 - 2\varepsilon)\|x\|_2^2 \int_0^{\varepsilon} e^{-2t\lambda} dt = \frac{1 - 2\varepsilon}{-2\lambda}(e^{-2\lambda\varepsilon} - 1)\|x\|_2^2,$$

This implies that

$$\|W(\tau)x\|_2 \geq \kappa\|x\|_2, \quad \text{with } \kappa^2 = \frac{1 - 3\varepsilon}{-2\lambda}(e^{-2\lambda\varepsilon} - 1) > 0. \qquad (15)$$

Now, we can apply Lemma 1, to conclude that $W(\tau)$ is invertible with bounded inverse and its norm satisfies the inequality $\|W(\tau)^{-1}\| \leq \kappa^{-1}$ for every $\tau > 0$, where κ is given by Equation (15), which is independent of τ. $\square$

Now, we are ready to complete the proof of Theorem 2.

Proof of Theorem 2. Define

$$u_0(t) = -e^{-tA^*} \cdot W^{-1}(\tau)x_0 \quad \text{for every} \quad x_0 \in \ell^2, \tau \in \mathbb{R}^+. \qquad (16)$$

We are going to show that the control function defined by (16) solves the control problem. By (12), for every fixed $\tau \in \mathbb{R}^+$, we have

$$-\int_0^\tau e^{-tA} u_0 dt = \int_0^\tau e^{-tA} e^{-tA^*} dt \cdot W^{-1}(\tau) x_0 = x_0. \tag{17}$$

It remains to show that u_0 is admissible, i.e., there exists $\tau > 0$ such that $\|u_0\| \le \rho$. For any $x_0 \in \ell^2$ with $\|x_0\|_2^2 \le \kappa^2 \rho^2$, then (17) implies that $x(\tau) = 0$ for the solution started from x_0. The definition of $W(\tau)$ (12) and (15) imply

$$\begin{aligned}
\int_0^\tau \|u_0(t)\|_2^2 dt &= \int_0^\tau \|e^{-tA^*} W^{-1}(\tau) x_0\|_2 dt \\
&= \int_0^\tau \left\langle e^{-tA} \cdot e^{-tA^*} W^{-1}(\tau) x_0, W^{-1}(\tau) x_0 \right\rangle dt \\
&= \langle x_0, W^{-1}(\tau) x_0 \rangle \le \|x_0\|_2 \cdot \|W^{-1}(\tau) x_0\|_2 \le \rho^2.
\end{aligned} \tag{18}$$

This proves local null controllability. To obtain estimates on the transition time, it is sufficient to obtain estimates on τ such that

$$\|x_0\|_2 \cdot \|W^{-1}(\tau) x_0\|_2 \le \rho^2.$$

This is equivalent to

$$\left(\frac{-2(1+\lambda)}{\kappa(e^{-2(1+\lambda)\tau} - 1)} \right)^{1/2} \|x_0\|_2^2 \le \rho^2,$$

which is satisfied if

$$\tau \ge \frac{\|x_0\|_2^4}{\kappa \rho^4} \ge \frac{1}{2|\lambda + 1|} \log \left(1 + \frac{2|\lambda + 1|}{\kappa} \frac{\|x_0\|_2^4}{\rho^4} \right).$$

This completes the proof of local null controllability.

We are going to consider the cases $\lambda < -1$ and $\lambda = -1$ separately to prove global null controllability. The proof is similar to that of [14], here we bring it for completeness.

Global null controllability for $\lambda < -1$. We will show that $\|W^{-1}(\tau) x_0\|_2 \to 0$ as $\tau \to +\infty$. In order to do so, we need to improve the inequality in Equation (14).

Exploiting Theorem 1 and invertibility of e^{tA^*} for any $x \in \ell^2$, we obtain that

$$\|e^{-tA^*} x\|_2 \ge \|x\|_2 \left(\|e^{tA^*}\| \right)^{-1} \ge e^{-t(1+\lambda)} \|x\|_2.$$

Consequently, for any $x \in \ell^2$ we can state that

$$\langle W(\tau) x, x \rangle = \int_0^\tau \|e^{-tA^*} x\|_2^2 dt \ge \int_0^\tau e^{-2t(1+\lambda)} \|x\|_2^2 dt = \|x\|_2^2 \cdot \frac{e^{-2(1+\lambda)\tau} - 1}{-2(1+\lambda)}.$$

We have $\|W^{-1}(\tau)\| \le \kappa^{-1}$ by lemmata 1 and 2. Therefore, we have

$$\kappa^{-1} \|x_0\|_2^2 \ge \langle W(\tau) W^{-1}(\tau) x_0, W^{-1}(\tau) x_0 \rangle \ge \|W^{-1}(\tau) x_0\|_2^2 \cdot \frac{e^{-2(1+\lambda)\tau} - 1}{-2(1+\lambda)}.$$

Thus,

$$\|W^{-1}(\tau) x_0\|_2 \le \left(\frac{-2(1+\lambda)}{\kappa(e^{-2(1+\lambda)\tau} - 1)} \right)^{1/2} \|x_0\|_2. \tag{19}$$

This implies $\|W^{-1}(\tau)x_0\|_2 \to 0$, since the right-hand side of the above inequality converges to 0 as $\tau \to +\infty$, Therefore, (18) implies that there exists a τ_0 such that

$$\int_0^\tau \|u_0(t)\|_2^2 dt \le \rho^2 \quad \text{for all} \quad \tau > \tau_0,$$

hence u_0 is admissible. This completes the proof for $\lambda < -1$.

Global null controllability for $\lambda = -1$. In this case, we do not have exponential stability of the trivial solution. Therefore, the above approach does not work apply in this case. In this case, we use local controllability and asymptotic stability of the trivial solutions. In particular, we set the control function 0 until the trajectory reaches a small neighbourhood of the origin and then use the control function given in Equation (16), which remains admissible in the vicinity of the origin. More precisely, if $\|x_0\|_2 \le \rho\kappa$, where κ is the constant defined in Equation (15), we can define the control function as

$$u_1(t) = -e^{-tA^*} \cdot W^{-1}(1)x_0 \quad \text{for every } x_0 \in \ell^2, \text{ with } \|x_0\|_2 \le \rho\kappa. \tag{20}$$

Then, using Equation (18), we obtain

$$\int_0^\tau \|u_1(t)\|_2^2 dt \le \|x_0\|_2^2 \cdot \|W^{-1}(1)\| \le \rho^2,$$

and

$$x(1) = e^A x_0 + e^A \int_0^1 e^{-sA} u(s) ds = 0.$$

Moreover, due to the stability of the system (1), there exists $\tau_0 = \tau(\kappa, \rho, x_0)$ such that $\|e^{tA}x_0\|_2 \le \rho\kappa$ for any $t \ge \tau_0$. Therefore, we set

$$u_0(t) = \begin{cases} 0, \text{ if } t \le \tau_0, \\ u_1(t), \tau_0 \le t \le \tau_0 + 1. \end{cases} \tag{21}$$

which is admissible and leads to $x(\tau_0 + 1) = 0$ for the corresponding solution of (3). This completes the proof.

No global null controllability for $\lambda > -1$. This follows from Example 1, since in this case, we have directions in which the solution grows exponentially fast. □

3.3. Differential Game of Pursuit in ℓ^2

In this section, we give proof of Theorem 3.

Proof. Theorem 2 implies that the equation

$$\langle x_0, W^{-1}(\tau)x_0 \rangle^{1/2} = (\rho - \sigma)^2$$

has a solution $t = \tau$.

Let $v(\cdot)$ be any admissible control of the evader. We define the strategy of the pursuer as follows:

$$u(t, v) = v(t) - e^{-tA^*} W^{-1}(\tau)x_0. \tag{22}$$

Notice that $e^{-tA^*} W^{-1}(\tau)x_0$ is the same as in (20), which shows the relation of the pursuers strategy to the optimal control problem. Below we show the admissibility of the strategy (22). First, by the definition, we have

$$\|u(t, v)\|_2 = \left(\int_0^\tau \|u(t, v)\|_2^2 dt \right)^{1/2} = \left(\int_0^\tau \|v(t) - e^{-tA^*} W^{-1}(\tau)x_0\|_2^2 dt \right)^{1/2}$$

Applying the Minkowski inequality on the right-hand side, we obtain

$$\|u(t,v)\|_2 \le \left(\int_0^\tau \|v(t)\|_2^2 dt\right)^{1/2} + \left(\int_0^\tau \|e^{-tA^*}W^{-1}(\tau)x_0\|_2^2 dt\right)^{1/2}$$

Finally, using the constraint on $v(\cdot)$, choice of τ and (22), we obtain

$$\|u(t,v)\|_2 \le \sigma + \left(\int_0^\tau \langle e^{-tA^*}W^{-1}(\tau)x_0, e^{-tA^*}W^{-1}(\tau)x_0\rangle dt\right)^{1/2}$$

$$= \sigma + \left(\int_0^\tau \langle e^{-tA}e^{-tA^*}W^{-1}(\tau)x_0, W^{-1}(\tau)x_0\rangle dt\right)^{1/2}$$

$$= \sigma + \langle x_0, W^{-1}(\tau)x_0\rangle^{1/2} = \sigma + \rho - \sigma = \rho.$$

Now, we check that the equality $x(\tau) = 0$ holds. Indeed, for

$$\xi(\tau) = x_0 + \int_0^\tau e^{-sA}(u(s) - v(s))ds = x_0 - \int_0^\tau e^{-sA}e^{-sA^*}W^{-1}(\tau)x_0 ds = 0$$

we have $x(\tau) = e^{\tau A}\xi(\tau) = 0$. This completes the proof. $\square$

4. Discussion of the Results and Further Questions

In this paper, we considered an infinite system of linear ODEs with a special operator $A = \lambda \operatorname{Id} + \sum_{i=1}^N a_i E^i$ on the right-hand side, where $a_i \ge 0$ and $\sum_{i=1}^N a_i = 1$. We established stability and controllability of the system when $\lambda \le -1$. Moreover, we consider a differential game with integral constraints and show that the pursuit game can be completed under natural conditions.

Our main motivation for this choice was to construct a toy model of linearly interacting points, whose behaviour was different than that of uncoupled system. Moreover, we study differential game problems.

It turns out our results are still true if the $a_i \ge 0$ and $\sum_{i=1}^N a_i = 1$ conditions are replaced with the following conditions: the polynomial $f(z) = a_1 + a_2 z + \ldots + a_N z^{N-1}$ is positive definite, but to keep the exposition simple, we did not pursue such a generalisation here. Here, we have the following open problems: it would be interesting to obtain similar results for a more general system of the form

$$\dot{\mathbf{y}} = A\,\mathbf{y} + Bu, \quad \mathbf{y}(0) = \mathbf{y}_0, \tag{23}$$

where $A: \ell^2 \to \ell^2$ is a bounded operator and $B: \mathcal{L} \to \mathcal{L}$ is an operator from a (possibly finite-dimensional) subspace $\mathcal{L}$ of ℓ^2. Our proofs suggest that if B is the identity operator and the spectrum of A lies on the left-hand side of the imaginary axis, then (23) is globally asymptotically stable. The invertibility of the Gramians also appears to work, since it is a perturbation argument. However, for global null controllability, we need new estimates for the inverses of the Gramians, or another approach altogether. For general B, the situation is unclear, and it would be interesting to obtain similar conditions to the classical Kalman condition (see, for example, [21] (Theorem 1.16)) or an analogue of the Fattorini–Hautus condition, but in both situations, the exact conditions are unclear. For the Kalman condition, injectivity of an operator is not sufficient for invertibility, and for Fattorini–Hautus, one typically assumes a countable spectrum with certain properties (see, for example, [7] and references therein).

Author Contributions: Conceptualization, I.Z., G.I., M.R. and T.C.; methodology, I.Z., M.R., G.I. and T.C.; investigation, I.Z., M.R., G.I. and T.C.; writing—review & editing, M.R. and T.C.; supervision, M.R. and T.C. All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The authors declare no conflict of interest.

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